

Algebra

The first records dealing with adding or subtracting the same magnitude on both sides of an equation are found in the Arabic writings of Al-Khowarizmi (Mohammed ben Musa, which means Mahomet, the son of Moses) about 830 A.D. This is an important work on which subsequent algebraic developments were based. The word algorithm is derived from the name of this ancient mathematician. Algorithm originally meant “the art of calculating.” In England mathematicians were called calculators. Now algorithm means calculating by any method following a given set of rules. Just as the definition of algorithm evolved, so does the concept of algebra continue to evolve. Originally, algebra contained no symbols. Its rules were proclaimed “as if they were divine revelations, which the reader was to accept and follow as a true believer” (Eves, 1990). Ironically, today’s mathematics student who does not fully absorb the concept being taught may feel there is a leap of faith involved in some algebraic calculations. There was a void created in the history of mathematics when the Greeks dropped algebraic proofs in favor of geometric language due to the Pythagoreans’ inability to deal with irrational numbers. Algebraic reasoning was resurrected by Al-Khowarizmi, which eventually led to the proof of the existence of irrational numbers, allowing mathematicians to once again embrace algebraic reasoning and carry on from the point where the Greeks had abandoned it.

Algebraic methods and notations have been improved and revised. Unlike arithmetic, where $3 + 4 = 7$ (written that way for bases 8 and larger), algebraic notations like $x + y = z$ take on different meanings in different contexts. Thus, the subject of algebra provides challenges for some individuals because of their difficulty in dealing with the abstractions associated with unknowns.

Descartes (about 1637 A.D.) used symbolic notation to express algebraic calculations. He also used letters at the beginning of the alphabet (a, b, c) to denote known quantities and letters from the end of the alphabet (x, y, z), particularly x, to indicate unknown quantities. He used numbers to indicate different powers of a quantity.

Individuals with skills limited to computational ability offer little to society mathematically. Technology can do the arithmetic. Society needs thinkers to employ technology. Still, in many algebra classrooms, students are not permitted to use technology until fact and operational mastery is evident. If a student cannot exhibit the skill of multiplying decimals at some satisfactory level, that student is prohibited from advancing algebraically until that computational ability can be mastered. How much better it would be to use a calculator and see what can be accomplished mathematically. There is a difference between arithmetic and mathematics, isn’t there?

Traditionalists say something like, “The mathematics I learned and the way I learned it was good enough for me, so it is good enough for today’s student.” Unfortunately, that statement is far from true. Yet the resistance to curricular and conceptual change is formidable. Today’s world is much more mathematical than

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yesterday's, in that productivity in today's world requires greater mathematical abilities than did yesterday's. Even most common percents, ratios, and discounts are done with calculators instead of by hand. Tomorrow's world will be even more mathematical than today's. As technological advancements continue, some segments of mathematics will decrease in importance whereas others will grow.

NCTM encourages the integration of problem-solving techniques throughout the curriculum. This is an outgrowth of complaints from industry about the inability of employees to interpret answers. An algebraic application of this position could be built around the assumption that a decision needs to be made about pay procedures in a company. Should they offer positions that are commission only, or should they provide a base salary supplemented by a commission? Basic algebra provides the answer, but often individuals struggle with the solution. Time is money, and if an employee wrestles too long with this dilemma, there is a negative impact on the productivity of the company.

WHO SHOULD LEARN ALGEBRA?

Most of today's mathematics education world presents the position that algebra should be learned by all students if they are to be functional contributors to the world of the future. This is based on suppositions like:

- The need for the development of a logical thought process
- The expected increase in the use of technology
- The need for employees to interpret professional literature

Motivating a student to learn algebra is a challenge. There are avenues of pursuit that will help relieve this predicament. One remedy could be the chart from "When are we ever gonna have to use this?" (Saunders, 1993). Topics are grouped by the subject areas of basic math/pre-algebra, beginning algebra, geometry, second-year algebra/trigonometry, and other topics (calculus, calculator use, computer use, problem solving, mathematical modeling, and so on). Beginning algebra topics listed are using formulas, linear equations, linear inequalities, operations with polynomials, factoring polynomials, rational expressions, coordinate graphing, linear systems, radicals, quadratic equations, and algebraic representation. Over 100 career options are listed on the chart. Career choices include all the mathematically obvious ones like engineer and scientist, but they also list trades like carpenter, electrician, mechanic, painter, and plumber. The medical professions are represented with categories like dentist, dietitian, doctor, nurse, physical therapist, veterinarian, and x-ray technician. Airline pilots, TV camera operator, museum curator, farmer, firefighter, golf pro, real estate agent, and waiter/waitress are also listed. As students ask for rationalizations about the need to learn algebraic concepts, the careers listed should convince them of the universal advantages of having a command of the subject.

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BEGINNING ALGEBRA

Are algebra I, beginning algebra, beginning algebra, introductory algebra, and pre-algebra all the same course? What should be taught in beginning algebra? Who should determine the concepts for a beginning algebra course? Many school districts define an algebra course as the material covered in 180 days, thereby implying that approximately 150 hours have been spent on algebra instruction (assuming 50-minute periods). In reality, teachers often hope for 120 hours of instruction.

Usiskin (1987, p. 428) presented the following as a jumping off point for his discussion on what should or should not be included in a beginning algebra course:

- Operations with positive and negative numbers; evaluation of expressions
- Solving of linear equations, linear inequalities, and proportions
- Age, digit, $d = rt$, work and mixture word problems
- Operations with polynomials and powers
- Factoring of trinomials, monomial factoring, special factors
- Simplification and operations with rational expressions
- Graphs and properties of graphs of lines
- Linear systems with two equations in two variables
- Simplification and operations with square roots
- Solving quadratic equations (by factoring and completing the square)

Many argue that we should

- Use applications rather than contrived word problems.

- Delete factoring trinomials (keep monomial factoring and special factored forms)

- Delete rational expressions requiring factoring

- Use the quadratic formula to solve quadratic expressions.

Beginning algebra is new for students. Prior to this point in their mathematical development, generalizations played a relatively minor role in the scheme of mathematics. Now, generalizations and processes occupy center stage in their mathematical learning. Variables are introduced formally, and notions from their arithmetic background are extended to the set of real numbers. Language becomes more formalized, and symbolic manipulation and its associated skills begin to be central themes of each student's mathematical existence. Unification and blending of topics and subjects begins to occur. Problem solving occupies a more central location, and expectations about systematic reasoning increase.

Beginning algebra begins to apply pressure to the mathematical framework of the student. For many, this is the initial exposure to mathematics beyond memorization, or a cookie-cutter-type curriculum. The course is a transition from the specifics of arithmetic into a confusing world where things are allowed to change. How these changes occur influences the final outcome. Before, they were told that division by zero was undefined. Now they are expected to be able

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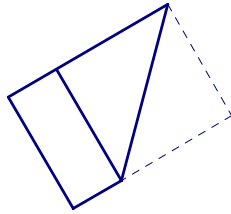
to extend that to the realization that if a denominator of a fraction is “ $X + 3.2$,” then X cannot be -3.2 because that would yield zero for the term. Such maneuvers are not easy for all students to see immediately.

Because the difference of two squares is so common in algebra, students have been encouraged to memorize this as one of the special factored forms, often with no concrete explanation about why the solution is as it is. How much better a concrete explanation would be! DO the following to see why

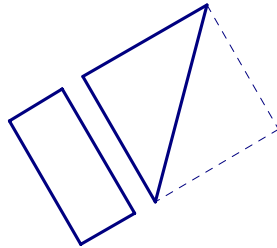
$$a^2 - b^2 = (a + b)(a - b)$$

Use any rectangularly shaped piece of paper

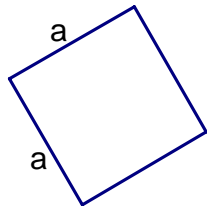
Fold it so a square can be cut from it



Cut the excess “tail” off



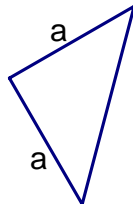
Express the side length of the square in terms of a variable.



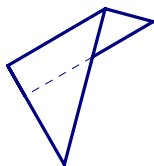
What is the area of the square?

$$a^2$$

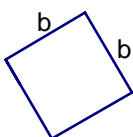
Fold the square along the previously established diagonal



Make a fold in the triangle so it is parallel to one of the legs of the triangle.



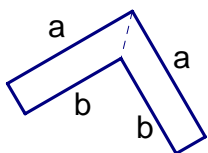
Cut along this new fold line and, for the time being, lay the resultant trapezoid to the side. Open the triangle and express the side length of the small square in terms of a variable.



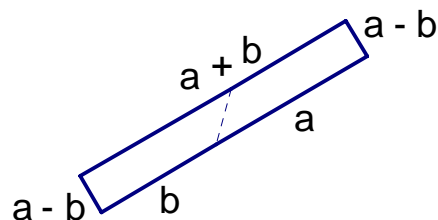
What is the area of the square?

$$b^2$$

Open the folded trapezoid to show an L-shaped concave hexagon.



Cut the "L" along the established diagonal. Flip and rotate one of the two trapezoids, placing the two together at the common diagonal to form a rectangle.



The area of this rectangle, in terms of its dimensions, is $(a + b)(a - b)$, which is the same as $a^2 - b^2$ since the area of the little square was subtracted from the area of the initial square.

Doing activities like "Squares," described here, assists students with the ability to visualize a process. Similar benefits can be derived as factoring skills are developed (Brumbaugh, 1994, p. 18).

VARIABLE

It is generally accepted that the concept of a variable is difficult for beginning algebra students to comprehend. They may have been exposed to the idea of a variable in a multitude of settings prior to coming to algebra class. They may or may not be aware of the times when they used a variable. In the early grades, they dealt with equations like $2 + 3 = \#$. Most probably they were not aware of the idea that the $\#$ stood for an answer they were looking for. Later, as students are exposed to word problems and formulas, variables appear in natural settings. For example, when they find the area of a rectangle, they use the formula $A = lw$, and

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it is accepted that “A” stands for “area,” “l” represents the length of the rectangle, and “w” is the width. As they do word problems and translate them into equations to be solved, words give way to letters that represent the words and the concept of variable appears again. These informal exposures to variables give way to more formal approaches in the beginnings of algebra. Students like to make the correlation of a variable to a meaningful concept like $A = \text{Area}$ and $l = \text{length}$ and often ask why we use “x” as the variable rather than “n,” which could stand for number, or “m” for missing number?

Once a variable is defined, typically the texts go into how to write variables and how to operate with them. One significant technological issue is raised at this point. Most textbooks define five times a number n to be either $5 \cdot n$ where the “dot” is elevated, $(5)(n)$, $5(n)$, or $5n$. Then the texts almost universally abandon all forms except for $5N$, which uses implicit multiplication. “Everyone knows that the multiplication symbol is there” expresses the common mentality on this issue. Students become accustomed to writing $5n$. The question is whether the student understands this notation to represent the product of “5” and “N.” When technology is used, many software versions require insertion of the multiplication symbol. Thus, part of the world of technology is significantly different from the written text world. Is that difference acceptable? If the difference is accepted, how is it explained to the students? Because we are now well into the technological age, we need to be prepared to explain to students why mathematical notation is not universal.

The multiplication symbol is an excellent place to introduce this topic. The * operator is a common computer symbol for mathematics. We could not use a period because of its many other uses in computer languages. Many students might ask, “What about the symbol x?” Computer languages are forced to recognize x as the English letter.

Once the definition of variable is established, the work usually focuses on substituting some number for that variable and evaluating the expression. Extended exposure includes other variables that are added, subtracted, multiplied, and divided, but no exponents are used with the variables. Students are asked to convert word phrases into algebraic expressions using variables with exercises like:

A number decreased by 4.

Some number m is 5 greater than 13.

A value m is tripled, then added to 86.7.

In the curricular continuum, exponents are generally one of the next places variables are encountered. In a setting like x^n , x is defined as the base and n is the exponent that “shows how many times x is used as a factor.” One question that should be asked eventually about the definition is, “What if the exponent is 0.5? How do we write x as a factor 0.5 times?” Exponents lead into scientific notation and a way to write very large or very small numbers.

MULTIPLYING A MONOMIAL AND A POLYNOMIAL

Products of a monomial and a polynomial assume previous exposure has included collecting like terms. Significant amounts of time are spent dealing with situations containing negative factors. The distributive property becomes an essential ingredient in the understanding of future explanations involving the product of two polynomials. Most of the time, the multiplication is expressed horizontally in forms like $5(2m^3 + 6m^2 - 7m - 8)$. The monomial eventually includes variables and exponents, and the polynomial may involve more than one variable. However, there are some advantages to showing the product in vertical format as well.

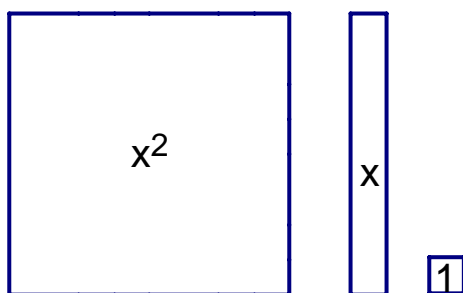
The idea of vertical multiplication is particularly advantageous if the students have had prior experience with expanded or partial product forms of multiplication. The product $3(21)$ would be written in partial product format as:

$$\begin{array}{r} 21 \\ \times 3 \\ \hline 63 \\ \hline \end{array}$$

3 from 3 times 1
60 from 3 times 20.

The 20 is significant in showing how place value is determined. The vertical writing is important in establishing format, because the partial products can be collapsed into the standard algorithm. It is meaningful to relate the current work to prior efforts as students are guided to a deeper level of understanding. Both formats are vital to helping students conceptualize different models of multiplication commonly used in algebra.

A set of manipulatives can be created that is similar to the base 10 blocks. These manipulatives are available commercially [Algebra Lab Gear (Creative);



Algeblocks (ETA)] but they can be made easily. Refer to the big square as x^2 , the rectangle will be named x , and the little square will be a unit or 1. Use the pieces to build rectangles. There are a few ground rules. It may appear that a number of units are the same length as an x if they are placed edge to edge in a straight line segment like square bricks in a wall. (In the commercial sets the units will not add up to an x or y . They just don't fit that way. The creators made sure of that!) It is not permitted to make trades in such an instance. Trades such as this were permitted with base 10 blocks, where ten ones could be traded for one ten or ten

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tens would be traded for one hundred. The reason these trades cannot be done with the algebra manipulatives is that x is a variable or unknown. If six units are traded for an x when the value of the x is really nine, errors would be produced. Thus, exchanges are not permitted between pieces in the set.

The set of algebra manipulatives can be used to express products in a manner very similar to that used with the base 10 blocks. $3(2x + 1)$ would be shown as three sets of two x s being added to a 1, giving a grand total of six x s and three 1s. The students would then interpret this product as $6x + 3$. Note the similarities between this and 63 expressed as $6(10^1) + 3(10^0)$ or $6(10) + 3$. The product needs careful association with the manipulatives to assist the students in creating a mental image of the product. As with young children in their early development of the concept of multiplication, care must be taken to not rush too quickly to the abstraction. Assure that appropriate intermediate steps are provided to assist the students in visualizing the overall operation. The students will want to shed the manipulatives as quickly as possible, which is fine, as long as they have had some time to understand the impact of the operation. The degree of difficulty of the problems can be expanded with manipulatives, but these examples can quickly become cumbersome.

Multiplication of a constant times a variable expression like $3(x^2 - 5x + 8)$ is easy to show. The size of the constant factor can make the setting overly complex because of the number of pieces needed to show the product. However, it is important to realize that even the large, cumbersome products can be shown - - they just take time to create. The ideal is to work with small products, which are easy to show, and have the students understand what is happening. Then abandon the manipulatives as quickly as possible with the perception that they can be recalled if needed. Establishing a solid foundation with the small, simple problems is crucial to the students' ability to handle more complex issues. Discarding the manipulatives too quickly is akin to just telling the students the process. That method is not effective. Ultimately, if the manipulatives are not permitted to run their full course, it is better to not use them at all.

Exponents greater than squares can be shown with the manipulatives. Creation of the tools becomes cumbersome. x^3 can be built with a normal three-dimensional approach to building a cube that is x units on an edge. These are represented in many of the commercial sets. Beyond that, representations become difficult. This amplifies the need to establish a solid foundation with the x , x^2 , and perhaps x^3 . The hope is that the underpinnings will be sufficient for the students to visualize the processes necessary to complete the assigned tasks.

THE PRODUCT OF TWO BINOMIALS

Most authorities agree there is value in having students understand the operations to be performed at some level beyond mechanical. Regrettably, most people still stress doing the process to get the answer, while overlooking a multitude of opportunities to connect the operation with previously covered topics and establishing groundwork for future study. Prior to working with the product of two binomials, students cover the products of two monomials and the product of a monomial and polynomials. F O I L (First, Outer, Inner, and Last) is a mnemonic used by many teachers to instruct students on how to find the product of two binomials. Let $(A + B)$ and $(C + D)$ be the two binomials to be multiplied.

Multiplying the *First* terms of each binomial gives AC

Multiplying the *Outer* terms of each binomial gives AD

Multiplying the *Inner* terms of each binomial gives BC

Multiplying the *Last* terms of each binomial gives BD.

Although FOIL accomplishes the task of having students find products of two binomials, little opportunity for understanding the process is evident. Use of the distributive property of multiplication over addition, even here, would give one rule that works for all polynomials. The major tragedy is that many students who learn to FOIL with limited understanding then proceed to apply this special case to polynomials, not just binomials.

If the idea of the distributive property of multiplication over addition on the set of real numbers was developed during exposures to the products of monomials with polynomials, then that idea can be extended to the product of two polynomials. Consider the product of two binomials $(x + 3)(2x + 4)$. Negatives need to be considered, but because of the potential difficulties involved in multiplication of signed values, it is advisable to avoid them until students begin to understand the situation. If the distributive property of multiplication over addition on the real numbers was used earlier, then the problem can be expressed as $x(2x + 4)$ and $+ 3(2x + 4)$ or $(x + 3)(2x)$ and $(x + 3)(4)$. Note that “+” $3(2x + 4)$ is purposely used here with the intent of laying groundwork for negative factors. Initially it is important to maintain order and use the commutative property of multiplication on the set of real numbers to change things if desired. Once this becomes a student’s reflex behavior, the formality can be de-emphasized. When the problem $(x + 3)(2x + 4)$ is seen as $x(2x + 4)$ and $+ 3(2x + 4)$ or $(x + 3)(2x)$ and $(x + 3)(4)$, reference can be made to prior work and the results compiled accordingly. If this relation is clearly made, then factoring of trinomials becomes an easier concept to grasp. The distributive property approach to the product of two binomials explains why FOIL works and provides a vehicle to be used when dealing with the products of polynomials with more than two terms.

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Partial product is a powerful connector for many students. If they have seen $(34)(21)$ expressed in partial product form, working with $(x + 3)(2x + 4)$ becomes almost intuitively obvious (assuming appropriate background work).

$\begin{array}{r} 34 \\ \times 21 \\ \hline 4 \text{ from } 1 \times 4 \\ 30 \text{ from } 1 \times 30 \\ 80 \text{ from } 20 \times 4 \\ \hline 600 \text{ from } 20 \times 30 \\ \hline 714 \end{array}$	$\begin{array}{r} x + 3 \\ \underline{2x + 4} \\ 12 \text{ from } 4 \times 3 \\ 4x \text{ from } 4 \times x \\ 6x \text{ from } 2x \times 3 \\ \hline x^2 \text{ from } x \times x \\ \hline x^2 + 4x + 6x + 12 \text{ or } x^2 + 10x + 12 \end{array}$
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This might be even more dramatic for some students if $(34)(21)$ is written in expanded form for the multiplication.

$\begin{array}{r} 30 + 4 \\ \times 20 + 1 \\ \hline 4 \text{ from } 1 \times 4 \\ 30 \text{ from } 1 \times 30 \\ 80 \text{ from } 20 \times 4 \\ \hline 600 \text{ from } 20 \times 30 \\ \hline 714 \end{array}$	$\begin{array}{r} x + 3 \\ \underline{2x + 4} \\ 12 \text{ from } 4 \times 3 \\ 4x \text{ from } 4 \times x \\ 6x \text{ from } 2x \times 3 \\ \hline x^2 \text{ from } x \times x \\ \hline x^2 + 4x + 6x + 12 \text{ or } x^2 + 10x + 12 \end{array}$
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TOPICS TO HELP VISUALIZE FACTORING

The algebra student will have been exposed to factoring in previous classes. Anytime factoring is done, essentially the area of a rectangle is given and the task is to determine the dimensions. This can be established as a part of the introductory review. The following shows where the prior exposure occurred and how to build on that information to deal with an algebraic setting.

Question: When multiplying, what names are given to the numbers?

Probable response: Numbers.

Better question: When multiplying, what are numbers called?

Probable response: Factors and product.

Question: Where is multiplication used in geometry?

Probable response: Finding area. Each time two numbers are multiplied, you are finding the area of a rectangle. When dividing, you are given the area of a rectangle and one dimension.

Question: What, then, is the task in division?

Probable response: To find the other dimension of the rectangle.

Question: What are other names for the length, width, and area?

Probable response: Factor, factor, and product.

Question: So if you have a product, what does it represent?

Probable response: The area of some rectangle.

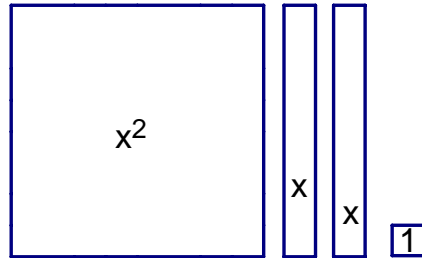
Question: What else can you determine if you have the product or area of a rectangle?

Probable response: Its dimensions.

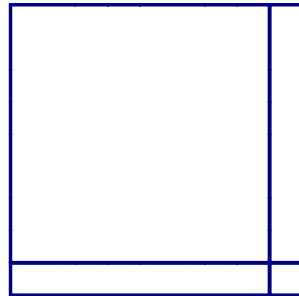
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Comment: Those ideas of dimensions and areas are used in algebra, too. We used that idea when we found the product of two binomials. Each factor was a dimension and the product was the “area” of a rectangle.

Question: You each have a set of manipulatives. If I had an x^2 , two x s, and a unit, can I build a rectangle?



Probable response: I can build a square, but all squares are rectangles.



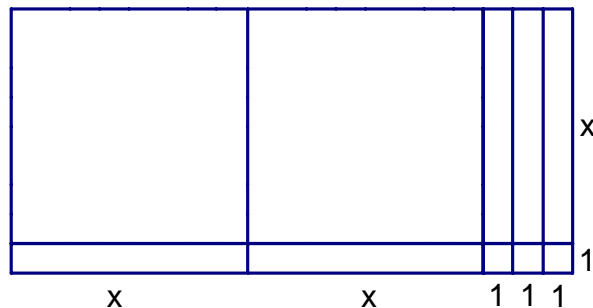
More examples would be given and the homework assignment should have the students using given sets of pieces to build rectangles. No mention is made of dimensions and all examples will result in a rectangle. For example, make a rectangle from two x^2 s, three x s, and two units; two x^2 s, five x s, and three 1s; and so on. Each time, the student is to sketch the solution using representations of the manipulatives.

The next lesson would begin by discussing the figures formed out of the given pictures, with students showing their sketches.

Teacher: Look at the picture made from two x^2 s, five x s, and three ones. What shape is the figure and what is its area?

Student: Rectangle and $2x^2 + 5x + 3$

Teacher: What are the dimensions of the rectangle?



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Student: $2x + 3$ long and $x + 1$ high.

Teacher: Think back to your basic multiplication. Is there more than one set of dimensions for the area of a rectangle?

Student: Maybe. For prime numbers the answer is no, but for composite numbers the answer is yes. For example, if the area is 12, the dimensions could be 1 by 12, 2 by 6, or 3 by 4.

Teacher: So a product might have more than one set of factors?

Student: Yes.

Teacher: Do you suppose that would be true with the algebraic expressions we have been dealing with?

At this point the students would be assigned the task of determining if the pieces could be arranged in more than one way to give a rectangle for some algebra problems, indicating whether or not there is more than one set of factors for a given product. The ensuing discussion would focus on the process and the algebraic name of factoring. The preceding vignette shows a series of questions, answers, and activities that can be constructed to assist students in creating mental images of the tasks they are asked to perform. This review provides connection to prior learning, refreshes basics, and establishes a beginning point for factoring.

DRILL IN BEGINNING ALGEBRA?

There is a need for some drill. The question becomes how much. Some insist on lists of problems to assure mastery of even the smallest nuances. Others say that as long as the student “really understands” the concept, there is little need for drill, because the students have mastered the process. Many adopt the middle-of-the-road approach.

Drill-and-practice can be made more enjoyable than it often is. Many number tricks have an algebraic base. For example, do the following problem and record the amount of time it takes you to get the answer.

1234567890

$$\frac{(1234567891)^2 - (1234567890)(1234567892)}{1234567890}$$

If you did this by performing the operations, even with technology, there is a quicker way. Algebraically, the problem is

$$\frac{x}{(x + 1)^2 - (x)(x + 2)}$$

Simplifying the denominator gives $x^2 + 2x + 1 - x^2 - 2x$, which is 1. The solution is the numerator of the original problem, 1234567890 (Brumbaugh, 1994, p. 7). Certainly, there is nothing wrong with doing the problem via the arithmetic route, but algebra is a lot quicker.

Students can easily be convinced of the need to “think” algebra when confronted with problems like the one used here and other “Pick a number” tricks.

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Pick any counting number.	x
Add the next highest counting number.	$x + x + 1$
Add 9 to the sum.	$x + x + 1 + 9$ or $2x + 10$
Divide this new sum by 2.	$\frac{2x + 10}{2} = x + 5$
Subtract 5.	$x + 5 - 5$
What did you get?	x
How does this work? Explain algebraically.	

Technology affords a multitude of opportunities for drill-and-practice. There is software available that essentially turns the computer into a glorified copy machine. In this mode, problem after problem is given to the student to work. For all practical purposes, the computer becomes an extension of the textbook in that a long list of problems is given to the student to practice. This appears to be a gross misuse of the power of the computer, and yet it is a popular mode because many students are enamored with the use of a computer. The computer could become an avenue in which the student wants to practice algebra skills.

TECHNOLOGY IN BEGINNING ALGEBRA

There are ways in which technology can be used for practice. A caution about technology first: Students must become aware of the fact that technology is only as good as the person using it. That is, an answer should not be accepted just because it was derived using technology. A skill that students should possess is estimation of values. Technology affords the opportunity to check the accuracy of those estimates. The use of a calculator to reinforce the concepts of exponents is an example. Does $8^9 = 9^8$? Are 9 factors of 8 equivalent to 8 factors of 9? The assistance of a calculator can help students develop a conceptual understanding of exponents quickly and easily.

Solving two equations in two unknowns is a typical activity found in beginning algebra courses. Most of the time, the first approach is graphing solutions, followed by substitution, addition/subtraction, and multiplication/division. Many text series devote an entire chapter, or at least several sections, to the treatment of these concepts, which means several days of instruction. Technology offers an alternative.

The following is a description of a few lessons dealing with solving two equations in two unknowns. The first demonstration dealt with graphing two equations and determining the point of intersection by zooming, tracing, and inspection using software. For homework the students were to graph pairs of equations and determine the coordinates of the point of intersection. Their results were to be sketched on paper for the next class.

One frequent question about using software to teach mathematics is whether or not the material being covered transfers to paper/pencil tasks because they are

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still the dominant mode used in testing settings. The answer is a resounding “yes.” In the class following the homework of sketching graphs of intersecting linear equations, the students were presented with the graphs of two intersecting lines ($y = x + 3$ and $y = -x + 7$) on the viewing screen and asked to determine the point of intersection. By inspection, they determined that the common point had coordinates (2, 5). The class was ready to proceed to the next topic.

The software used for this lesson permitted storage and recall of equations. The first equation, $y = x + 3$, had been stored in compartment A and $y = -x + 7$ was placed in B. Investigating the impact of operating on the stored equations led to a discovery. We looked at $A + B$, which gave “ $y = x + 3 + y = -x + 7$.” Simplifying gave “ $y + y = x + 3 - x + 7$,” leading to the observation that all the y s were on the left and all the x s and constants were on the right. Another simplification resulted in “ $2y = 10$ ” and one student said that because that was one equation in one unknown, the y value would be 5. As this statement was being made, several students vocalized that this was the same as the y value obtained when graphing that pair of equations. A summary of what had transpired followed, and the class went to the computer lab to work with a group of equations (all coefficients of variables were positive or negative one) that had been stored. Their task was to experiment and arrive at some conclusions. They could add or subtract the stored equations, dealing with two at a time.

The ensuing discussion focused on realizations like:

Adding two equations where the respective signs of the variables were the same resulted in one equation in two unknowns.

Subtracting two equations where the respective signs of the variables were the same resulted in a nonsense situation like $0 = 6$.

When the signs of the respective variables were opposite and the equations were added, something like $0 = 6$ appeared.

If the signs of the respective variables were opposite and the two equations were subtracted, one equation in two unknowns appeared.

If one of the variables had the same sign and the other had opposite signs, either addition or subtraction would yield an equation in one unknown, which could then be solved.

Once the solution was determined, it could then be substituted into one of the original equations, and that situation solved for the value of the other unknown. After a short discussion of their perceptions, the class appeared to have a solid grasp of the general impact of adding and subtracting equations.

At this point the students were asked how they would deal with a situation like “ $y = x + 3$ ” and “ $2y = 3x + 4$.” Comments were quick and incisive, particularly considering that they had just done addition and subtraction. The basic statements made by the class were:

It won't do any good to add or subtract them because you will get one equation in two unknowns.

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If we had $2y$ we could subtract.

That means we have to multiply that one equation.

Yes, but don't forget to multiply both sides.

Then we can subtract one equation from the other and get one equation in one unknown.

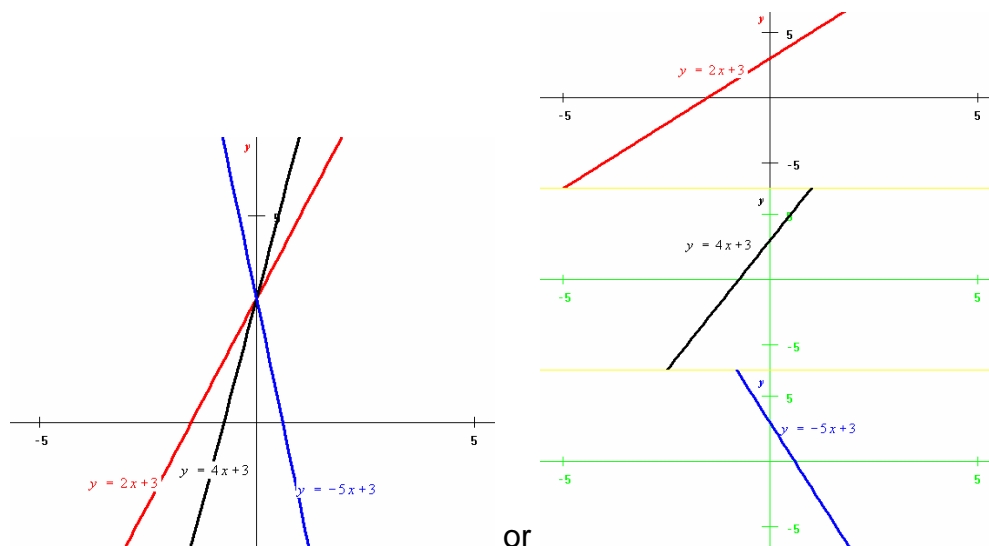
Maybe we should multiply by a negative 2 because we make fewer sign mistakes when adding as opposed to subtracting.

At this point, the class was sent back to stored equations on the computers. These equations had integral coefficients. In a short time the class defined most of the rules for multiplication, division, and substitution as a means for solving two equations in two unknowns.

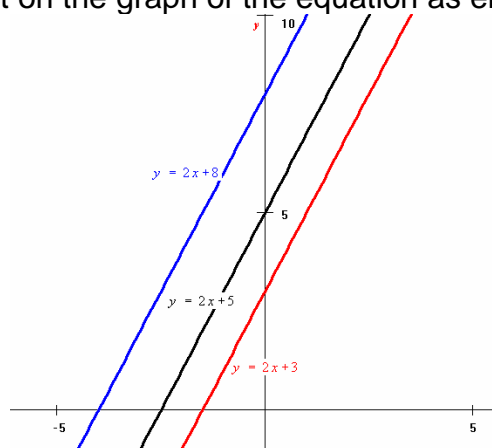
All of the activities described since the review of solving two equations in two unknowns were completed in less than an hour. The students were able to transfer their computer work to paper-and-pencil assignments and complete them satisfactorily. Their retention was good. They were able to apply the knowledge throughout the rest of the class. Most important, what often takes several weeks to accomplish in a typical classroom setting that does not use technology as a teaching/learning tool was completed in a few hours, and the students' attitude about solving two equations was positive.

In the preceding discussion about solving two equations in two unknowns, technology was used as a discovery tool. There is ample opportunity for students to discover basic concepts with technology. Consider learning about the impact of changing the coefficient of x in a linear equation written in slope intercept form. Traditionally, the teacher leads a discussion that guides students to the appropriate conclusions. During the course of that discussion, different equations will be sketched with varying degrees of precision. That same lesson delivered with technology can be much more dynamic, assuming the teacher is working with a projected image of a linear equation. For the sake of this discussion, it is assumed that the class is familiar with the use of technology by the teacher. Thus, it is safe to assume they will ask what happens if the 2 in $y = 2x + 3$ is changed to 4. Then, what happens if the 4 in $y = 4x + 3$ is changed to -5. And so forth. It does not take long for the students to conclude that as the coefficient of x increases, the line gets steeper. They have just described the concept of slope and the definition could now easily be formalized. Certainly, this could be done without technology, but the speed, precision, spontaneity, and flexibility would be lacking.

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It is possible that some student would want to change the constant. Careful discussion on the part of the teacher can lead to a delay of that idea until the slope exchange is completed. Eventually both the slope and the constant would be changed, but at that point the students should be adept at predicting the impact on the graph of the equation as either is altered. The advantage of



technology is that it assists the students in creating mental images of what is happening. This, in turn, strengthens their understandings and provides stronger foundations for future work. Equally significant is that the students become willing to ask “What if” questions, something that will be invaluable in helping the student become self-motivated lifelong learners.

One excellent opportunity to integrate mathematics and science is through the use of probes that can be used to gather data (IBM Personal Science Laboratory, and probes for the CFX-9850GB Plus and TI 8X). The following activities describe one possible use of the probes.

Using the distance sensor, a series of activities can be used to build the concept of the slope of a line. Students often deal in a world of absolutes. That is, they are confident they can stand perfectly still. Have a student stand in front of the

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distance probe assigned with the task of standing “perfectly” still for a few seconds while their distance from the probe is measured. As the experiment is conducted, a horizontal (or so it seems) line appears on the screen. Magnification of the graph shows that it is not a horizontal line but may vary by as much as 5 centimeters. The students are amazed about this but soon come to realize that they do make small movements; their clothing will move because of breezes in the room, they will make slight reflexive body movements because of heartbeat or breathing, and so forth. After this discussion, rescale the screen so the small variations are not evident.

Ask a student to walk at a constant rate away from the probe. It usually takes a few tries before a relatively straight graph line is established. This line will be used as a reference as another student is asked to walk, making a line parallel to the one on the screen. Even though they may not be able to formally express it, most students know the meaning of parallel. As this is attempted, comments from the kibitzers in the class will encourage the walker to go faster or slower. The comments begin the understanding of slope. The term slope may not be used, but comparative statements like one line is steeper than the other will be common. This is the beginning, and the class can be guided to a formal definition of slope from here. This discussion can be extended to the difference between parallel lines. Proper calibration of the software can have the lines intersecting the y-axis at different points and yet the lines will be parallel. Prior to this activity, the students learned about the slope intercept form of the line. Here, or at any point in the discussion, the mathematical characteristics could be discussed in the detail necessary to accomplish the objectives of the lesson.

One extension of the line-walking activity can stimulate some interesting and informative conversation and learning. A line is established, and a student is assigned the task of walking a line perpendicular to it. Before long the class will determine that the student needs to walk toward the probe if the initial line was established by walking away from the probe. In most instances the same walking rate will not work. Once a line that appears perpendicular is walked, the equations of the two can be investigated. Repetition of the experiment a few times should lead to equation pairs that have slopes that are close to being multiplicative inverses of each other and their signs will be opposite. Out of that, the class can determine the definition of slopes of perpendicular lines.

At this point, revisit standing still in front of the probe. After a horizontal line is established, ask if it is possible to walk a vertical line. Attempts at acting this out in front of the probe can become quite lively. Students will jump, duck, move their hands as fast as they can, group together and move quickly, place several books in front of the probe, each held by a different student, move the books quickly out of range, and so on. These attempts will not result in a vertical line. However, the students are honing in on the idea that they must move instantaneously if they are to walk a vertical line. Soon they will conclude that it is impossible for them to be transported from one location in front of the probe to another without some

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change in time. Thus, it becomes impossible to walk a line perpendicular to horizontal, and the concept of an undefined slope for a vertical line is established.

APPLICATIONS OF ALGEBRA

Research shows that activities can have a positive impact on the attitudes of students. Questions like “When will I ever use this?” are common. A likely translation of questions such as that is, “Where can I find an application of this concept in my world today?” It is imperative that the world be viewed from the perspective of a secondary student at this point, not from the perspective of an adult trying to convince the student to accept what is being given. There are situations that show algebra basics being used in the real world, as demonstrated by the following dialogue between a student and a tutor, which led to an activity called “Speed Trap” (Hynes & Brumbaugh, 1976, pp. 40–42).

Student: Why do I need to put in steps when solving something like $2x = 6$? Everybody knows the answer is 3.

Teacher: True, but you are learning a process. What if you had something like $14.5267y = 53.79825$? You wouldn't know the answer to that one. You learn the process in $2x = 6$ so you can solve things like this.

Student: OK, but where would you get something like that?

Teacher: Let's talk about law enforcement officers catching speeders. Modern technology allows a car to be timed as it travels a known distance. If the car goes through the distance too fast, the driver is exceeding the posted speed limit and, more than likely, that driver is going to have an opportunity to meet a representative of a police unit. We will go to a local road and establish a speed trap to see the equations that are generated.

A variety of other things will need to be done to complete this activity. They include:

Measure a reasonable distance, marking the beginning and end to establish the speed trap (100 yards provides enough time lapse to decrease the impact of many time measurement errors) Calculate the minimum legal time to cover the trap distance.

Establish the car part used to indicate entrance and exit of the trap (the front bumper does fine).

Determine how entrance and exit will be signaled to the timer (raising and lowering hands will work).

Station an individual at each end of the speed trap so the timer can see both of them.

Clock cars going through the trap.

Knowledge of speeders will be immediately evident from the recorded time. At this point, the students need to determine how much over the legal speed the car was traveling. In order to answer this question, the students need to compute the speed using the known time and distance. They are now solving equations that

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they will not reflexively know the answer to. In the process, they have also seen an application of the concept being discussed in class.

ALGEBRA IN PATTERNING SITUATIONS

In the Problem Solving Workshop, a fast way to find the sum of consecutive counting numbers was discussed. Gauss, who, as an elementary student, quickly found the sum of the first 100 consecutive counting numbers to be 5,050 by using something similar to the following process.

$$\begin{array}{cccccccc} 1 & + & 2 & + & 3 & + & \dots & + & 98 & + & 99 & + & 100 \\ \hline \frac{100}{101} & + & \frac{99}{101} & + & \frac{98}{101} & + & \dots & + & \frac{3}{101} & + & \frac{2}{101} & + & \frac{1}{101} \end{array}$$

Because there are one hundred “101s”, the sum would be $(100)(101)$. But this sum is twice what it should be because each addend was used twice. The sum of the first one hundred consecutive counting numbers is $\frac{(100)(101)}{2}$. In general,

the sum of the first n consecutive counting numbers can be found by using $\frac{(n)(n - 1)}{2}$. Most generalizations are going to require algebraic skills and

notations in their final expressions. One such example involves the idea of being paid a penny on the first day, two cents on the second, and each subsequent day finds the pay to be double that of the preceding day. Typically, the discussion

Day	Pay For Day	TOTAL Pay
1	1	1
2	2	3
3	4	7
4	8	15
5	16	31
n	2^{n-1}	$2^n - 1$

focuses on how much money is earned on any given day, and that is often generalized because of the pattern. However, another generalization can be derived from the problem. If the pay was doubled for each of 30 days, starting with a penny on the first day, the pay for the 30th day would be \$5,638,709.12 and the total payment for all 30 days would be \$10,737,418.23. Without the assistance of patterning and algebraic skills, the solution to the question would be difficult to obtain (but not impossible). The idea of patterning and some basic algebra, coupled with the power of technology, puts a problem such as this within the reach of a wide selection of students.

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Number theory relies on patterns that frequently can be generalized, calling in the use of algebra. Consider

$$1 + 2 = 3$$

$$4 + 5 + 6 = 7 + 8$$

$$9 + 10 + 11 + 12 = 13 + 14 + 15$$

$$16 + 17 + 18 + 19 + 20 = 21 + 22 + 23 + 24$$

These equations offer a plethora of pattern and algebraic opportunities. In this instance several observations are possible. The first equation uses the first three counting numbers. The second equation uses the next five counting numbers, the next seven, and so on. Each equation begins with a perfect square, which makes sense if the students have been exposed to the task of finding the sum of consecutive odd counting numbers beginning with 1. ($1 + 3 = 4$; $4 + 5 = 9$ [where $4 = 1 + 3$]; $9 + 7 = 16$ [where $9 = 4 + 5$ and $4 = 1 + 3$]; and so on.) Even beginners to patterning will soon notice that the first addend in each equation is a perfect square, algebraically expressed as n^2 where n represents the row number. A little prompting should lead to the conclusion that there are n addends on the right of the equal sign and $n + 1$ to the left. Given that information, a student should be able to describe any row from the table of numbers given.

THE ROLE OF ALGEBRA IN PROOF

Most non-geometric proofs rely heavily on algebraic skills. The concept has roots in beginning patterns, perhaps as simple as getting the next counting number. Young children often get to the next counting number in their exposures by adding one more object to a set of elements that comprise the number they just learned. That is, once a child masters “fourness” (four objects can be recognized in any configuration), five is presented. Often the presentation involves showing a set of five things, which is discussed and manipulated until “fiveness” becomes a part of the world of that child. The concept builds on the idea of one more than the last, and even though it is not expressed algebraically, the foundations are there. We would say that is just “ $x + 1$ ” where x represents the last number the child mastered.

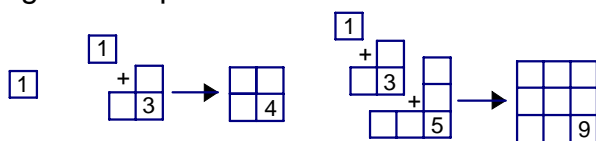
As students progress through their learning exposures in mathematics, the complexities of the settings increase and the concept of proof begins to evolve. Young children see a set of three objects and a set of two objects placed together to form a set of five objects. They take the same sets of two and three and get five. The reversal of cardinalities is significant because eventually the setting is summarized into problem pairs of $2 + 3 = ?$ and $3 + 2 = ?$. At this point, the child gets an initial exposure to the commutative property of addition on the set of counting numbers. Eventually that becomes generalized to the commutative property of addition on some given set like the real numbers expressed as $a + b = b + a$. These formative abstractions are important proof building blocks.

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Once the abstractions are started, more formalized expressions of proof become possible. The ability to represent things in algebraic terms is helpful at this stage of development. For example, students might conclude that the pattern

$$\begin{aligned} 1 &= 1 \\ 1 + 3 &= 4 \\ 1 + 3 + 5 &= 9 \\ 1 + 3 + 5 + 7 &= 16 \\ &\text{etc.} \end{aligned}$$

will generate a list of perfect squares, which could be generalized algebraically, leading to the need for algebraic capabilities. This then becomes the beginnings of proof. At the appropriate level, the information in the table can be expressed by: "The sum of the first n consecutive odd counting numbers is n^2 ." One nice advantage to this pattern is that it can be shown concretely.



Pairing this generalization with that from the discussion about Gauss' generalization of the sum of the first n consecutive counting numbers to be $\frac{n(n+1)}{2}$ leads to a wonderful opportunity for a generalization that turns out to be

false. We have:

$$\frac{n(n+1)}{2} = \text{the sum of the first } n \text{ consecutive counting numbers}$$

$$n^2 = \text{the sum of the first } N \text{ consecutive odd counting numbers}$$

It would seem reasonable that $\frac{n(n+1)}{2} - n^2$ would yield the sum of the first n consecutive even counting numbers. Assuming that and simplifying, gives

$$\begin{aligned} \frac{n(n+1)}{2} - n^2 &= \frac{(n^2 + n) - 2n^2}{2} \\ &= \frac{-2n^2 + n}{2} \end{aligned} \text{ , which is negative and cannot possibly}$$

represent the sum of the first n even counting numbers.

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Many algebra texts contain a “proof” that $2 = 1$. It is presented here to show you another example of the need to convince students to pay close attention to each step and detail in a proof.

Let $A = B$	A and B are any real numbers
$A^2 = AB$	Multiply both sides of equation by same value
$A^2 - B^2 = AB - B^2$	Subtract same value from both sides of equation
$(A + B)(A - B) = (A - B)B$	Factor both sides of equation
$\frac{(A + B)(A - B)}{(A - B)} = \frac{(A - B)B}{(A - B)}$	Divide both sides of equation by same value
$A + B = B$	Result of division
$B + B = B$	Substituting B for A because $A = B$
$2B = B$	Addition
$2 = 1$	Divide both sides of equation by same value

This result contradicts what is known to be true and yet students often take a long time to recognize that because $A = B$, they cannot perform the step involving dividing by $A - B$.

Number tricks offer a wonderful opportunity to lead students into the world of algebraic proof. Suppose 9 and 5 are two of the three digits in a three-digit number. You are to find a third digit so that an addition equation can be formed where both addends and the sum are permutations of the same three-digit number comprised of 9, 5, and the third digit you find. This problem could be solved using guess-and-check routines, which some teachers use to generate the need for proof expressed at an abstract level. However, an algebraic approach shows some advantages. Let n be the third digit used to make the three-digit numbers to solve this problem. The six possible combinations that can occur are $95n$, $9n5$, $59n$, $5n9$, $n95$, and $n59$. Neither of the numbers with 9 in the hundreds place can be an addend because that would force regrouping so the sum would be a four-digit number. Assume that the sum must begin with 9. Assume one of the addends begins with 5. $9n5 - 59n$ is one possibility for the other addend. Expanding gives

$$\begin{aligned}900 + 10n + 5 - (500 + 90 + n) &= 100n + 50 + 9 \\400 + 9n - 85 &= 100n + 50 + 9 \\91n &= 256 \\n &= 2.813 \text{ (not digit)}\end{aligned}$$

Another possibility would be $95n - n59 = n95$

$$\begin{aligned}900 + 50 + n - (100n + 50 + 9) &= 100n + 90 + 5 \\950 - 99n - 59 &= 100n + 90 + 5 \\796 &= 199n \\4 &= n\end{aligned}$$

Here is another demonstration of the power of algebra in determining why something works (or can be proved). Investigate a different algorithm for multiplication where the ones digits of the two factors sum to 10 and all other digits of the two factors are duplicated (143×147 or 52×58). Find the product of the ones digits. That product becomes the ones and tens digit in the answer. In

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52×58 it would be 16 because of 2×8 . Call all of the digits to the left of the ones digit Z. Note that in each example the value to the left of the ones digit is the same. Multiplying Z and $(Z + 1)$ will provide the rest of the answer. In 143×147 , that would become 14×15 . This product is placed to the left of the product of the ones digits. To show why this works, let $10A + B$ be the first factor and $10A + (10 - B)$ be the second.

$$\begin{aligned}(10A + B)(10A + (10 - B)) &= 100A^2 + 100A - 10AB + 10AB + B(10 - B) \\ &= 100(A^2 + A) + B(10 - B)\end{aligned}$$

The number of opportunities to employ algebra as an investigation tool in proof is ample enough that, with a little energy on your part, each student can be shown the power of this branch of mathematics.

TUTORIALS FOR LEARNING ALGEBRA

One effective way to help students break the algebra barrier is the use of tutorials. Many schools have determined the importance of incorporating the graphing calculator and have found the funds to provide this technology for each student in the classroom. New graphing calculator technology such as Casio's FX-2.0 hand-held computer algebra system has made algebraic tutorials affordable for each student to have in the mathematics classroom.

The FX-2.0 combines the features of a fully functional graphing calculator and computer algebra system with the power of a computer algebra tutorial. All through the algebra curriculum, students learn the importance of solving linear equations. The use of the FX-2.0 algebra tutorial can allow students to visually see the outcome of their mathematical procedures as they are performed on the tool. For example, when trying to solve the equation $\frac{3x + 2}{4} = 12$, students

frequently decide that they should subtract two from both sides of the equation. The FX-2.0 allows the student to enter the equation as shown. If the student wants to subtract 2 from each side of the equation, the student presses the subtraction key followed by 2 and the execute key. The result would be:

$$\frac{3x + 2}{4} - 2 = 12 - 2, \text{ which shows the student that 2 is being subtracted}$$

from both sides of the equation. The FX-2.0 performs the operation that the student entered. Now the student can simplify the equation to view the result of the performed operation, which would be $\frac{3x}{4} - \frac{3}{2} = 10$.

The student can quickly see that the performed process has left two fractions rather than one. They might conclude they are not as well off as they were with the original problem statement. With the push of a button, the student can return to the original equation and then multiply by four on each side of the equation,

$$\text{giving } \left(\frac{3x + 2}{4}\right)(4) = (12)(4).$$

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Simplifying the equation, the student can see $3x + 2 = 48$ and should realize that the solution seems closer now. Subtracting two, followed by dividing both sides of the equation by three gives the solution.

Another powerful feature of the FX-2.0 is the capability to provide tutorial assistance when solving equations. The student can press the “next” function and the tutorial will respond by displaying the next step for solving the equation. This allows the student to learn by observing a step-by-step solution process. The FX-2.0 will perform the instruction in English, Spanish, French, German, or Italian!

WORD PROBLEMS IN ALGEBRA

Word problems are an integral part of mathematics in general and beginning algebra in particular. Students need to deal with word problems as they begin working with real-world applications of the concepts they are studying in mathematics. The word problems they encounter need to be from their perspective, not from that of an adult. The word problems should appeal to a student, and it is not always sufficient to assume that the text problems will hold sufficient attraction for the students. In fact, in many instances, the text problems hold little appeal for students.

Frequently textbook word problems are grouped by problem type. That is, a section in a book will deal with rate, percent, mixture, or age problems. In the respective sections, the problem type is considered almost exclusively. Generally they apply some newly studied principle. It is common for a section of mixture problems to follow exposure to a study of percents in beginning algebra. The mixture problems will deal with applications from the real world of chemistry and industry where solutions of a mixture are given and the desire is to change the content of one ingredient. Often a procedure is established for the student to follow as the problem is worked. These procedures are beneficial to completing the assigned task of doing a mixture problem, but are they damaging in that the student is almost programmed to follow a routine. The reinforcement comes because the assigned problems are built around teaching the student the routine with, perhaps, some subtle alterations. Essentially, they become specialists in a given problem type for the time it is being studied. After the test is given, another problem type may be discussed and developed. After a few days of dealing with the second problem type, the first is forgotten. Thus, the way we approach word problems can handicap students as they attempt to apply their mathematical skills in real-world settings.

ALGEBRA IN THE K–20 CURRICULUM

“Algebra for all is the right goal at the right time. We just need to find the right algebra.” was the feeling generated at a conference in 1997 (Brumbaugh, 1997). The hope was to build a coherent vision of algebra for Grades K–14, and then channel energies so the end product would be consistent.

Look at old algebra books. There has been little change in them over the past 150 years. Certainly color and pictures have been added. There is a lot more white space on the page. Some have cartoons. Some contain historical notes. Some describe uses of the content in the real world. These are all good changes. But look beyond them to the guts of the text. You see a static system. You see things being taught the same way, year after year. Until the 1980s the sequence was pretty much unaltered. A little bit of technology is inserted. There are some pedagogical changes. Still, most kids see little value to learning algebra, a subject that is often viewed as the gatekeeper of quantitative literature.

For sure, algebra needs to be livened up. We need to convince students that there is value in learning it. We need show them more examples of its use in their world. We need to think about how the whole subject should be structured, presented, applied, and sold to the students. Algebra needs to be approached for four different perspectives: graphical, tabular, symbolic, and verbal. The students need to have the ability to move between these different venues. They need to learn how to symbolize situations that have a numerical basis, structure that circumstance to show what is going on, and then model it. To help our students do this, we must think of how they transition from arithmetic to rate based structures. Then we have to help them generalize things into abstract algebraic models. Finally we need to develop the idea of x varying and its impact on $f(x)$, and how the two are hooked together.

Algebra is the ability to pull structure from a problem, then manipulate and interpret what we see. Students need to be involved with data and number early on. They need to learn to look for and describe patterns. Quantity needs to be recognized in what they discuss. Perhaps we should consider studying functions before equations. Evaluate what has been covered. Consider the meaning of the content. Then go to the abstract models.

Technology cannot be overlooked as we think of algebra. We must make some decisions on technology for doing algebra as opposed to technology for thinking algebra. Technology can be used as a sledgehammer/answer generator. Hopefully that is not the role it takes on. Rather it is a tool to remove computational distractions, reduce dependence on mechanical algorithms, provide alternative methods of approaching problems, and develop communication skills dealing with methods of solving problems. The focus is switched from getting answers to explaining processes used and interpretation of results.

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Some say technology generates negative forces. They would say technology reduces exposure to proof but we say it raises questions. Detractors of technology would say it creates dependence on machines rather than their own minds. Anti-technologists claim motivation is lost in the time required to learn the technology or that it is cheating. Of course the ability of all students to own the technology is raised. While that objection does carry some weight, it is not sufficient to stop the technological momentum. There are too many ways available to supply the technology if it is truly desired. Technology does not reason for the students. It is a tool. It can be used to force thought.

Algebra for all does not mean lowering expectations. It means raising expectations and expecting all students to meet them. It means rethinking the content and approaches to it. Each student must be able to access the important ideas of algebra. Each student needs to be able to experience algebraic concepts in a meaningful way. Each student has to make sense out of algebra. That is our responsibility as teachers of mathematics. We all have an obligation to deliver algebra in a way that will open doors for our students, not close them.

CONCLUSION

Algebra is viewed as the gatekeeper. Success in algebra opens the doors to additional mathematical study and a wide variety of careers. Lack of success in algebra can inhibit the number of opportunities available for an individual to pursue. This is but another amplification of the need for your students to do well in algebra. We have given you but a few indicators of the many ways topics within algebra could be approached in your classroom.

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