Sometimes students use an incorrect method or algorithm for solving a problem. Error patterns can be generated from lack of understanding of the rule, or a multitude of other reasons. Some errors are easy to determine. In this workshop, the emphasis is on diagnosing what error has been made and how to correct it.

If the subtraction problem 823 - 169 yields a response of 746, you are fairly safe in assuming the student subtracted the smaller digit from the larger in each place, paying no attention to whether the big digit was on the top or the bottom. This is a common error pattern. The basis for this error is often comes from the student being told that a “big number cannot be subtracted from a little one.” The genesis of this statement comes from the desire to have students realize the need for regrouping in the subtraction process.

Other error patterns are not as easy to determine. A student did the following examples. There was no scratch work. You see everything the student showed.

8431	9243	7152	8230
- 2576	- 1678	- 2649	- 4128
3965	5675	3513	3112

Try to determine the error being made before reading on. What was that student doing with these problems? How do we fix this?

Focus on 8431 – 2576 from the set of problems above. It appears as if subtraction facts are under control. Assuming that, the flaw must lie within the application of the algorithm. Regrouping is performed when necessary so that is not the problem. The trouble is associated with the regrouping. Where is the regrouping performed? The student is going to the leftmost digit to regroup, skipping any places between the location of the desired regrouping and that digit. In the first problem, the necessary “11” is created in the units place and the “8” is decreased to “7” in the thousands place. The tens-place subtraction decreases that new “7” in the thousands place to a “6” creating the necessary “13” in the tens place. The hundreds changes the new “6” a “5” and creates the necessary “14” in the hundreds place. Now, the problem becomes 5 thousand minus 3 thousand and it is done. That error is not easy to determine.

Although we cannot be certain, some guesses can be made as to the cause of this error. In this particular case, the class had most recently reviewed subtraction problems of the type 800 – 372 (discussed in the *Skills In Teaching Mathematics 2006 Workshop*), which required initial regrouping from the hundreds digit of the sum. It is assumed that the student was combining the two procedures and did some “creative mathematics.” Discussions with the student did not reveal a clear picture of the cause.

The preceding discussion should give insight into how this particular error pattern was diagnosed and remedied. Development of the skills necessary to perform such diagnosis and prescription is rooted in your understanding of the mathematics involved, time, and experience. Achieving the ability to determine error patterns is not an easy task, but it can reap large benefits for your students.

What is the error pattern for these problems?

36	45	92	68
× 47	× 68	× 63	× 98
492	640	276	964
204	420	602	1172
2532	4840	6296	12684

Try to figure it out before reading on. Otherwise you will read the solution, which will help you with this error type but will do little to build your diagnostic skills. It takes practice. Notice that, in the third problem from the left, the product of 92 and 3 is correct. This gives a partial clue. It appears as if the student has command of the multiplication facts but that the algorithm is confused. The student is adding the regrouping and then multiplying, as opposed to multiplying first and then adding the regrouped value. Look at the left problem and focus on 36 x 7. The student multiplied 7 x 6 and got 42, writing the 2 and carrying the 4. Then, the student added the carried 4 tens to the existing 3 tens in 36, getting 7 tens, which are then multiplied by the 7 ones from 47, giving 49, which is recorded to the left of the 2 already placed. The error involves commuting correct steps but it has a dramatic impact.

As your error pattern diagnostic skills increase, the problem areas are easier to detect. It does take conscious thought and practice. Multiple examples are necessary to enable you to define patterns.

ASSESSMENT

We in the mathematics community are fortunate to have the leadership provided by organizations like NCTM, MAA, American Mathematical Association of Two-Year Colleges (AMATYC), and MSEB. Their research and support have provided a solid background and set of guidelines we can use in the classroom. The Assessment Standards (NCTM, 1995b) provide a beginning point from which to work. You should be familiar with the recommendations provided in the publication. The discussion here provides general comments about assessment.

Assessment is more than paper–pencil testing. You look at the work students do and determine their strengths and weaknesses. You examine your teaching method to decide how it impacts the learning styles of your students. You look at the curriculum from the vantage point of the big picture. You assess your mathematics program in order to determine:

- The success of the overall program
- Whether students are learning
- How well the established mathematical goals are met
- If students can apply the mathematics in other curricula and life

When students are enticed to study more mathematics
The worthiness and usefulness of the content

As a teacher of mathematics, one of your basic goals is to help all students learn and enjoy mathematics to the fullest possible extent. Teachers assess achievement of students in the classroom because they must:

Determine the progress of each student
Ascertain the status of each student
Know the extent to which content and skills are mastered

Knowing the extent to which content and skills are mastered involves diagnosing strengths and weaknesses for each student. This is necessary in order to:

Accurately place students in the curriculum continuum
Assign grades
Help you learn how to teach more effectively
Gather specific rather than global information on individuals
Analyze how an answer is determined
Structure your teaching style

TESTING AND DIAGNOSIS

If one question is asked, you have little certainty about whether or not a student has mastered the material. Asking two questions dealing with the concept is better, but how sure can you be? If a student gets one of the two right, what do you know? You could give another test, assess other work the student has done, or talk with the student about the issue, but those each take time. Multiply the time required by the number of times you possibly will need to do something like this times the number of students you will be dealing with and you begin to see some constraints.

Maybe asking three questions would do it. If a student gets all three right or all three wrong, you would be fairly certain about the ability level pertaining to this concept. Perhaps five is a better number of questions to ask on a given concept. Your confidence level would be much greater if a student got all five correct. The problem is, the test becomes extremely long very quickly with five questions per concept.

Test length is particularly significant if concepts are closely and finely defined. Solving $X + 7 = 12$ involves a different set of skills from solving $3X + 7 = 12$. Realistically, even $X - 7 = 12$ is different from $X + 7 = 12$ in light of the fact that students, particularly weaker ones, frequently see these as entirely different, unrelated problem types. If you treat these as different problem types, and if you ask five questions per concept on a diagnostic test, the test will be very long. How much time can realistically be allocated to diagnostic testing?

Consider the following question that could be used to build part of a diagnostic test. Each response is generated by an error pattern like those mentioned earlier in this workshop. Some of the ways the problem could be done incorrectly are shown. There are others, but these will suffice for this discussion.

Step in solving	**Reason for the step**
3X + 5 = 17	The problem
- 5 = - 5	- constant from both sides
3X = 12	simplify
X = 4	÷ by coefficient
Another "solution"	
3X + 5 = 17	The problem
3X = 17 + 5	+ constant to both sides
3X = 22	simplify
X = 7	÷ by coefficient
Another "solution"	
3X + 5 = 17	The problem
3X = 17 - 5	- constant from both sides
3X = 12	simplify
X = 36	× by coefficient
Another "solution"	
3X + 5 = 17	The problem
3X + 5 - 5 = 17	- constant from left side
3X = 17	simplify
X = 5	÷ by coefficient
Another "solution"	
3X + 5 = 17	The problem
3X + 5 - 5 = 17	- constant from left side
3X = 17	simplify
X = 51	× by coefficient

The different "solutions" for 3X + 5 = 17 will be used to show how diagnostic testing and multiple choice items can be integrated to provide a lot of information. Good testing practices dictate that in multiple-choice tests, the same letter should not always represent the correct answer, but that will be ignored with the intent of eliminating one level of complexity as this idea is discussed. In this discussion, response A will be used for the correct response in each question in the table; response B for adding the constant to both sides (+ constant); C for subtracting the constant from both sides but then multiplying by the coefficient (- constant/X); D for subtracting the constant from the left side only and then dividing both sides by the coefficient (- left/÷); and E for subtracting the constant from the left side of the equation and multiplying the right by the coefficient (- left/X).

Answer	A	B	C	D	E
Question	Correct	+ con	- con, x	- left, ÷	- left, x
3M + 5 = 17	M = 4	$M = 7\frac{1}{3}$	M = 36	$M = 5\frac{2}{3}$	M = 51
2N + 4 = 18	N = 7	N = 11	N = 28	N = 9	N = 36
4P + 6 = 32	P = 6.5	P = 9.5	P = 104	P = 8	P = 128
7R + 3 = 12	$R = \frac{9}{7}$	$R = \frac{15}{7}$	R = 63	$R = \frac{12}{7}$	R = 84
2T + 5 = 13	T = 4	T = 9	T= 16	T = 6.5	T = 26

There are several poor testing practices shown in the table:

Some choices expressed as fractions and some as decimals

Responses not arranged in ascending or descending order

Inconsistent use of mixed numbers, improper fractions, decimals.

The desire here is to emphasize diagnostic matters, and other issues are ignored for the sake of this discussion. Someone selecting response A on all five questions in the test would adequately demonstrate mastery (assuming honesty). Choosing A in four out of five of the questions would probably be considered as having mastered the concept. As the number of correct responses decreases, the confidence level in the student's ability to correctly do that problem type also decreases.

Suppose a student selects B as the correct response for each of the problems in the table. Not only do you know the student missed them, but you also have a good idea about what was done to get those answers. This makes prescription easy and, in most cases, effective. A personal comment or note on the paper stating the error and what should have been done might correct the situation. This discussion would be the same for options C, D, or E. Only when a student responds randomly would there be a need for additional time invested to analyze the error pattern.

Using exercises like 7R + 3 = 12 in the table, tests only one concept. A different concept set is tested with exercises like 3X - 5 = 17 because other skills are involved. This could lead to a very long test. Test length can be resolved by giving many short tests. The time required to create such questions is not an insignificant factor. Suppose you decide to ask five questions like the ones in the table for each concept covered and that you teach the same course each year. A file of questions can be developed over a few years. In the first year you create five questions. The second year you create three more and use two of the initial five. In the third year, create two more and select three from the pool of eight developed earlier. At this point the amount of time required to create questions is drastically reduced, and you still have all the benefits of having a powerful diagnostic tool that provides information quickly. At this point, little time is required to evaluate student responses. Not only can you tell if the answer is correct but, in the event a student answers incorrectly, you have sufficient information to quickly determine the errors being made in most instances. Other testing formats should be considered; however, using a multiple-choice diagnostic instrument as a mainstay theme of your assessment program does have advantages.

In addition to formats different from multiple-choice, other assessment procedures help too. You need to listen, observe, reflect, interpret, and analyze your actions as well as those of the students as individuals and as a collective whole. Observation, checklists, criterion-referenced tests, norm-referenced tests, standardized tests, portfolios, journals, and so on all can contribute to the whole picture of the student's mathematical learning progress.

Assessment is a continuing integral aspect of teaching mathematics. It helps you determine if the students are learning what you think they should. It helps the student know if the ideas garnered are those that the teacher deems valuable. Assessment involves appraisal as well as measurement. No one form is appropriate for all aspects of assessment. You need to be aware of the multitude of avenues available to you and select those that will prove most beneficial to you in assisting your students to learn and appreciate as much mathematics as possible for each one of them.

SCORING RUBRICS

Rubrics are popular now, but most teachers of mathematics have used them on some sort of an informal basis for years. Rubrics demand that attention to be given to more than the answer in a mathematics exercise, which is a good thing. Certainly the answer is a significant factor that cannot be overlooked. However, there are a lot of little things that could go wrong, and in the process, generate a wrong answer. There is another question that begs to be asked when a student submits work - - is the process understood and the incorrect solution a result of a careless error?

Consider the error pattern discussion in this workshop and the accompanying exercises and you will see a multitude of situations where the process, rule, algorithm, concept, and so on, are not understood. Contrast that with the fact that a student may consistently show the product of 7 and 8 to be 54 rather than 56. If the student uses this incorrect fact and does everything else in the problem correctly, the answer will be wrong. The student clearly understands the concepts involved in the problem, but if you focus only on the answer, you get an very poor reading of the student's overall conceptualization of the setting.

Thus the plea that you employ a scoring rubric that considers crucial steps in solving the problem as well as the answer. Reconsider the earlier situation:

Step in solving	**Reason for the step**
3X + 5 = 17	The problem
- 5 = - 5	- constant from both sides
3X = 12	simplify

X = 4 ÷ by coefficient

The realization that the 5 is to be subtracted is significant and could be worth one point on a 5 point scale for the value of this problem. Another point could be awarded for the realization that the 5 is to be subtracted from both sides of the equation to keep things balanced. A third point could be awarded for the realization of the need to then divide by 3, and a fourth point for being aware that the division needs to be done on both sides of the equation, again to keep things balanced. The final point could be for the correct answer.

Yes, that rubric takes time to develop and score against as opposed with the idea of the right or wrong answer. But, the outcome for the student and your awareness of what that student actually knows about the mathematics of the situation is radically different when the rubric is used.

One last rubric point needs to be made. Suppose the student forgets to divide both sides of the equation in this discussion by 3 and gives an answer of 12. That is incorrect so a point is lost. Another point would be lost because of not dividing both sides of the equation by 3. Thus, the student would get 3 points out of a possible 5 for the problem, and in some arenas, that would be a passing score. That opens a whole new discussion about when a student should or should not pass. That discussion is beyond the scope of this workshop, but it should not be used as an excuse to revert to saying the problem is either right or wrong with no consideration to the steps involved and a scoring rubric.

Sidebar - - an easy way to help students remember that 7 x 8 = 56 rather than 54 is to write the equation as 56 = 7 x 8, where the sequencing of the 5, 6, 7, and 8 make the situation easier to remember.

REFERENCES

Ashlock, R. (2005). *Error Patterns in Computation.* Upper Saddle River, NJ. Prentice Hall.

Brown, J. S., Collins, A., & Duguid, R. (1989, January/February). Situated cognition and the culture of learning. Educational Researcher, 18, 32–42.

Brumbaugh, D. K., Ortiz, E., Gresham, G. (2006). *Teaching Middle School Mathematics*. Mahwah, NJ: Lawrence Erlbaum Associates.

Brumbaugh, D., Rock, D. (2001). *Scratch Your Brain C1*. Pacific Grove, CA: Critical Thinking Books and Software.

Clayton, M. (1999, November 9). Learning, colleges & universities. The calculator effect. Christian Science Monitor.

Cobb, R., & Steffe, L. P. (1983). The constructivist researcher as teacher and model builder. Journal for Research in Mathematics Education, 14(2), 82–94.

Crosswhite, J. F. (1986). Better teaching, better mathematics: Are they enough? Arithmetic Teacher, 34(2), 54–59.

Cummings, F. (1995). Equity in reforming mathematics and science education. The Common Denominator, 1(3), 1–2.

Davidson, N. (1985). Small-group learning and teaching in mathematics: A selective review of the research. In R. Slavin, S. Sharan, S. Kagan, R. Lazarowitz, C. Webb, & R. Schmuck (Eds.), Learning to cooperate, cooperating to learn. New York: Plenum.

Driscoll, M. R. (1994). Psychology of learning for instruction. Needham Heights, MA: Allyn & Bacon.

Esler, W. K., & Sciortino, P. (1991). Methods for teaching: An overview of current practices. Raleigh, NC: Contemporary.

Froelich, G. W., Bartkovich, K. G., & Forrester, P. A. (1991). Curriculum and evaluation standards for school mathematics, Addenda series, grades K–12, Connecting mathematics. Reston, VA: NCTM.

Grouws, D. A. (1992). Handbook of research on mathematics teaching and learning. New York: Macmillan.

Johnson, D. W. (1981). Effects of cooperative, competitive and individualistic goal structures on achievement: A meta-analysis. Psychological Bulletin, 89, 47–62.

Leach, E. L. (1992). An alternative form of evaluation that complies with NCTM Standards. Mathematics Teacher, 85(8), 628–632.

Mathematical Sciences Education Board. (1989). Everybody counts. Washington, DC: National Academy Press.

Mathematical Sciences Education Board. (1991). Counting on you. Washington, DC: National Academy Press.

Mathematical Sciences Education Board. (1993). Measuring what counts: A conceptual guide for mathematics assessment. Washington, DC: National Academy Press.

Mathematics education: Wellspring of U.S. industrial strength. Report of the Wellspring Symposium. (1988, December 15–16). Irvine, CA: ERIC Document Reproduction Service No. ED 338 498)

McKnight, C. C., Crosswhite, J. F., Dossey, J. A., Kifer, E., Swafford, J. O., Travers, K. J., & Cooney, T. J. (1989). The underachieving curriculum. Champaign, IL: Stipes Publishing Company.

Mullis, I. V. S., Dossey, J. A., Owen, E. H., & Phillips, G. W. (1991). The statement of mathematics achievement NAEP's 1990 assessment of the nation and the total assessment of the states. Washington, DC: U.S. Government Printing Office.

National Council of Teachers of Mathematics. (1989). Curriculum and evaluation standards for school mathematics. Reston, VA: Author.

National Council of Teachers of Mathematics. (1991). Professional standards for teaching mathematics. Reston, VA: Author.

National Council of Teachers of Mathematics. (1995a). Addenda series. Reston, VA: Author.

National Council of Teachers of Mathematics. (1995b). Assessment standards for teaching mathematics. Reston, VA: Author.

National Council of Teachers of Mathematics. (1995c, January). NCTM News Bulletin. Reston, VA: Author.

National Council of Teachers of Mathematics. (2000). Principles and standards for school mathematics. Reston, VA: Author.

National Research Council. (1990). Reshaping school mathematics: A philosophy and framework for curriculum. Washington, DC: National Academy Press.

Perkins, D. N. (1991, September). What constructivism demands of the learner. Educational Researcher, 58, 19–21.

Saunders, H. (1991). When are we ever gonna have to use this? Palo Alto, CA: Dale Seymour.

Saxon, J., & Hake, S. (1992). Math 76: An incremental developmental. Norman, OK: Author.

Shanker, A. (1995, July). A. F.T. decries lack of standards, “Gateway” exam. Education Week, 7.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. Educational Researcher, 15(2), 4–14.

Steffe, L. P., & Killion, K. (1986, July). Mathematics teaching: A specification in a constructionist frame of reference. In L. Burton & C. Hoyles (Eds.), Proceedings of the Tenth International Conference, Psychology of Mathematics Education (pp. 207–216). London: University of London Institute of Education.

U.S. Department of Education. (1991). America 2000: An education strategy. Washington, DC: Author. (ERIC Document Reproduction Service No. ED 327 009)