It is not enough for a child to have mathematical knowledge. They must have Mathematical Power to succeed. Mathematical power is the ability to feel comfortable in using mathematical knowledge to solve problems, to use mathematics in the real world, and to be willing to “try” and not feel afraid to fail.

RESEARCH IN GEOMETRY LEARNING

Diane van Hiele-Geldof and her husband Pierre van Hiele both did doctoral dissertations in 1984 that dealt with students learning geometry. Both dissertations dealt with the van Hiele assessment tool, which consists of five levels: visualization, analysis, informal deduction, formal deduction, and rigor. The van Hieles contended students can move from one category to another via appropriate experiences. Each of the five levels could be defined as follows, going from lowest to highest:

Visualization (Level 0) - - Students are aware of space. Geometric shapes are recognized holistically by their appearance without paying attention to component parts. Students functioning at this level can recognize geometric shapes and can reproduce them upon request. These students recognize squares and rectangles, but do not realize the presence of right angles, opposite sides of the same length, and so forth.

Analysis (Level 1) - - Students begin analysis of geometric concepts. Parts of geometric figures are recognized. Generally, definitions are repeated but not understood. Relations between properties are not explained. Students would be able to conclude that opposite angles of a parallelogram are congruent. They may not believe a figure can belong to more than one general class. For example, they might accept that a square is a quadrilateral, but they might resist the idea that that same square is also a parallelogram or rectangle, or both.

Informal deduction (Level 2) - - Definitions now make sense. Informal arguments about why things are as they appear begin to be formulated. Students know there are relations between properties of a figure. For instance, if opposite sides of a quadrilateral are congruent and parallel, the figure must be a parallelogram. They also become aware of connections between groups of figures, like all squares are rectangles, but not all rectangles are squares. These students know there is a collection of rules and axioms, but they cannot put them together via deductive techniques yet. They can follow formal proofs, but the logic of connections is not fully understood. Changing the order of steps in a proof different ways confuses them. These students are essentially unable to construct an original proof when starting with different material.

Formal deduction (Level 3) - - Students understand the role of axioms, rules, terms, theorems, definitions, and how they are interwoven. The ability to construct, not just memorize, proofs emerges. Doing a proof more than one way is within the sphere of these students. This means they are ready to study geometry as a formal mathematical system. As a part of that study, they will be able to write formal proofs using “if–then” type logic.

Rigor (Level 4) - - Abstractions are comprehended. Students can investigate and compare different geometries.

Essentially, there are specific questions and observations that will lead teachers to know where students are. Out of all the work that has been done, one dramatic point relates to the high school geometry curriculum: Most geometry course expectations have been set for students who are van Hiele Level 3 individuals. However, investigation indicates that most students entering high the course are van Hiele Level 2. That means one of two things: Either the student is doomed to failure or the course is watered down to allow students to succeed.

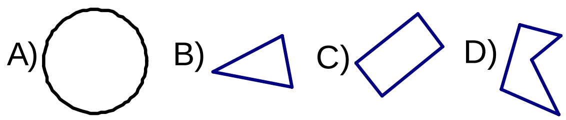
Piaget’s research indicates a different outline of how geometric reasoning and proof develop. Piaget contended that logical operations develop in individuals independent of the content in which they are working. These operations, according to Piaget, can be applied in a variety of settings and are used to establish new mathematical knowledge. A student who knows that both squares and rectangles contain only right angles, and that their respective opposite sides are congruent and parallel, along with the idea that a square also has adjacent sides congruent, could deduce that all squares are rectangles. This would be an application of the Piaget idea in which known information is used to create new information.

The van Hiele and Piaget thinking represent two major positions on the learning of geometry. Research can be gathered to support either posture, as is the case in so many things. In reality, the situation is not one where either belief is totally correct. More than likely, there is some combination that adequately reflects how most students learn geometry.

GEOMETRY IN THE ELEMENTARY SCHOOL

Before reading the next paragraph, please do the following. Take out a sheet of paper and draw a rectangle on it. After you have done that, read on.

Children enter elementary school having been exposed to a variety of shapes and related geometric concepts. “Standard position” of a figure usually means that one side of the figure is drawn parallel to the bottom of the board or page. For example, most of the rectangles students are exposed to are in standard position. Teachers draw them that way. Books show them that way. In other subjects, as well as in mathematics, pictures and special ideas are often presented in rectangular-shaped boxes that are in standard position. Now, consider a multiple-choice question where the student is to select one option: Which of the following is a rectangle:

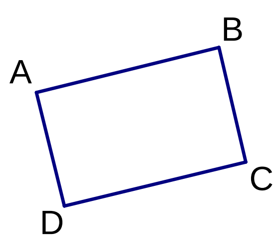


E) None of these

The number of students who select option (E) in this example is amazing. Why would they might select (E)?

Before reading on, color the rectangle you drew. Was your rectangle in standard position? When you colored your rectangle, did you fill the inside? That is not the rectangle! That is the rectangular region. The rectangle is the set of line segments that comprise the border of the figure.

Compare the ratio of the long side length to the short for the rectangle you drew.



Frequently the result will be such that, $\frac{\overline{BC}}{\overline{AB}} = 0.6180339\ldots$, the golden ratio, or golden section.

The reason you were asked to draw and color a rectangle is to emphasize the problem faced in the teaching of mathematics. Even individuals who have had several college level mathematics courses often draw the rectangle in standard position and color the interior. If you did that, you may have a tendency to not be mathematically precise. If we, as teachers of mathematics, are not accurate, how can we expect our students to be?

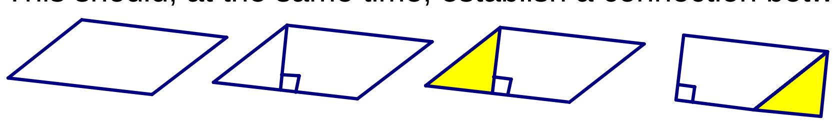
Current elementary education majors have a much broader exposure to mathematics than did their predecessors. Even with all the changes, the possibility of exposure to geometry is woefully lacking. Yes, geometry is taught in the elementary school. Line segments, rays, lines, angles, shapes, perimeter, and area are integrated throughout the curriculum. Shapes appear in other subjects in the form of boxes to highlight information, pyramids or arrowheads in history, playing fields or areas in physical education, and so forth. Textbooks include a wide variety of exposures for the students. However, the topic may appear at the end of a year when it may get less than adequate coverage.

There is geometry software that provides the opportunity to experiment and stimulate curiosity. Producers like Key Curriculum Press (Geometer's Sketchpad), Texas Instruments (Cabri), and others are creating collections of ideas and applications appropriate for students.

GEOMETRY TAUGHT IN THE MIDDLE SCHOOL

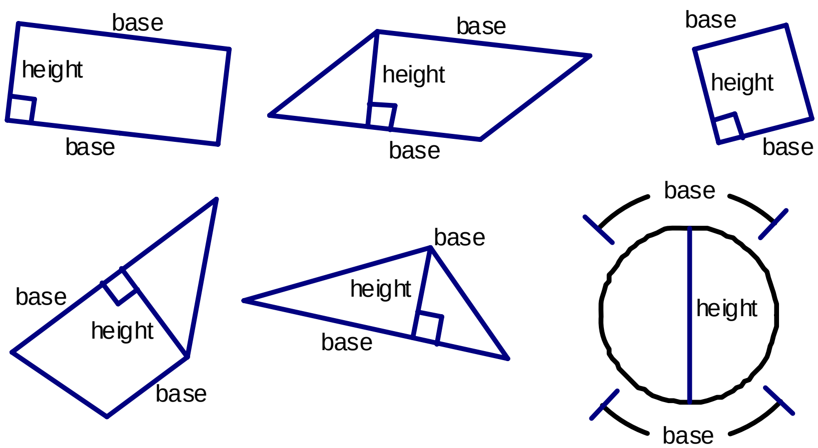
Geometry is scattered throughout the middle school curriculum. But what is taught and where or when? There is no specific course, and as the trend moves toward integrated topics, even if the course exists, it would be blended into a set of topics covered over the middle school years. So, what can be done in the middle school geometry class? Any topics introduced in the elementary curriculum can be extended in the middle school. Triangles, for example, can be revisited, this time looking at the measure of all angles, leading to the discovery that the sum of all interior angles of the triangle is 180°. That can be extended two ways. Investigations could focus on exterior angles and the sum of exterior angles of a triangle. This blends geometry with algebra, discovery, and generalization. Geometry should not be an unfamiliar topic in the 10th grade. Intuitive backgrounds can be established for a variety of topics.

There is a set of steps for showing how a parallelogram can be transformed to a rectangle, leading to the conclusion that the area of a parallelogram is base times height. This should, at the same time, establish a connection between two



shapes and how to find area. It should also raise a question. Because the new figure looks like a rectangle, why don't we use length and width as elements of the formula for the area of a parallelogram? Or, reverse the wording and use base times height for the area of the rectangle? It may appear as no big concern for us because we know, but students now have two more vocabulary words to learn and are identifying the same thing by two different names, each of which is to be used in a given setting (length with rectangle and base with parallelogram). Does that make sense? Does it really matter? Could we be more consistent in our discussion with students?

Establishing a connection between two shapes and finding area should not be abandoned too quickly. The formula typically taught for finding the area of a trapezoid is $\left(\frac{b_1 + b_2}{2}\right)(h)$. This formula can be written in a variety of formats and, depending on the algebraic skills of the students, confusion can exist as teachers attempt to shift from one form of the formula to another. For this discussion, use the form shown here and state it as the average of the bases, times the height. This form can be used for finding the area for a rectangle, parallelogram, square, trapezoid, triangle, and circle:



Even though the figures are not in standard position, you can see that the area for each of them can be found by taking the average of the bases times the height.

The rectangle and parallelogram transformations are relatively straightforward. For square, the area is usually given as $A = s^2$. Using that idea and the trapezoid formula of the average of the bases, the area of a square becomes

$$\left(\frac{\text{side} + \text{side}}{2}\right)(\text{side}) = \left(\frac{\cancel{2}(\text{side})}{\cancel{2}}\right)(\text{side}) = (\text{side})^2 = s^2.$$

The triangle seems a little confusing at first, but the lower base is the side to which the altitude is drawn. The upper base is the vertex, which is the top of the altitude, and has a length of zero. So for the triangle area, using the average of the bases,

$$\left(\frac{\text{base} + \text{zero}}{2}\right)(\text{height}) = \left(\frac{\text{base}}{2}\right)(\text{height}) = \frac{(\text{base})(\text{height})}{2}.$$

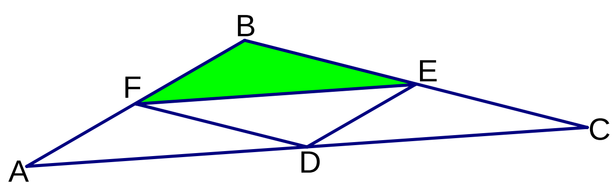
The circle area with this “one formula fits all” approach seems strange initially, and it does require some editorial liberty to discuss. We normally think of bases as being straight line segments. For this discussion, the base is a curved line segment, the length of which is one fourth of the circumference of the circle. The diameter, or height in this case, will be 2r. Using the average of the bases formula,

$$\left(\frac{0.5\pi r + 0.5\pi r}{2}\right)(2r) = \left(\frac{\pi r}{\cancel{2}}\right)(\cancel{2}r) = (\pi r)(r) = \pi r^2.$$

You might be saying, “Why didn’t someone show me this before?” “Why don’t we teach this method in the schools?” Those are legitimate questions that can be partially answered, but not totally. The easy part of the answer involves preparation to get to the level where a student understands the average of the bases formula. Some students are confused by the algebra involved and are unable to readily relate average of the bases to $(b_1 + b_2)\left(\frac{h}{2}\right)$ or $\frac{1}{2}(b_1 + b_2)(h)$.

Given the algebraic struggles that could exist, it would be difficult for students to understand applying that formula to a variety of shapes. Once the areas of rectangles and parallelograms are done using the trapezoid formula, some would question the reasonableness of returning to the “normal” formula. The hard part of the question about why we do not teach this “one formula fits all” approach in the schools is not so readily answered. It seems that this approach would be a wonderful extension for those students who have mastered the formulas for area of the shapes mentioned. However, this average of the bases formula is rather obscure.

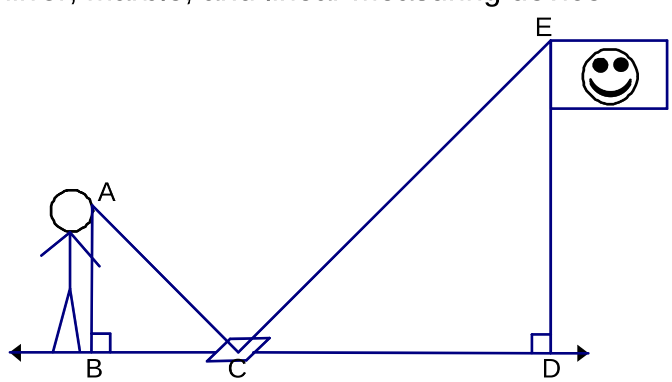
The middle school provides the setting for presenting a multitude of topics at an intuitive level that can be investigated in greater depth and more formally later. Joining the midpoints of the sides of a triangle to form four smaller triangles,



all of which are congruent and similar to the initial triangle. Here, triangles ADF, FEB, DCE, and EFD are congruent and similar to triangle ABC. Software provides students with the opportunity to investigate and develop insight into relations that exist in triangles. For example, they could conclude that the area of triangle ADF is 0.25 times the area of triangle ABC. They should also realize that segments AB and DE are parallel, and that the length of segment BC is twice that of segment DF. All of these intuitive feelings would spring from simple investigations and discoveries using the technology. Later, in a formal geometry class where such things are proven, the groundwork laid via the technology should establish valuable background information, and perhaps even a realization for the need for a more formal authentication of the intuitive feelings. Perhaps the Side Side Side congruence theorem would be used to establish that the four smaller triangles are, in fact, congruent.

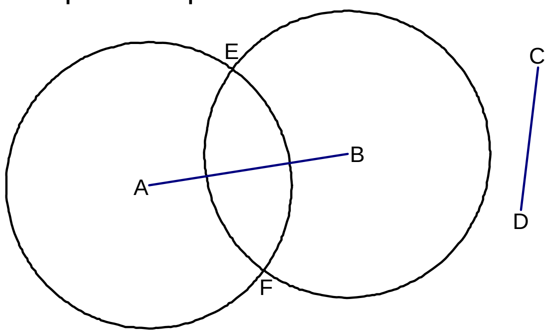
Another interesting extension involving the initial triangle ABC in the preceding paragraph can be developed. Segment DE joins the midpoints of sides AC and BC, respectively. Rather than using the midpoint, establish D on AC somewhere. Construct a line parallel to AB through D and create E as the intersection of the new line and side BC. Measure the length of segments AC and CD and the area of triangles CDE and CAB. Establish a ratio between the long and short length and the large and small area. When the length ratio is 2:1, the area ratio will be 4:1. When the length ratio is 3:1, the area ratio will be 9:1. Before long, the students should be able to generalize the pattern. At the same time, many students will become aware of the differences between linear changes and those of areas. The informal setting leads to an extension that could provide stimulation and a desire for formalization for some students.

Extending the idea of similar triangles that stimulated the last two ideas, it is important that students not only gain a feel for the existence of such things, but also that they see uses of them. In geometry, use of similar triangles to measure heights is common, but typically they occur in the form of pictures in the text. One activity that could be used in measuring heights of inaccessible objects involves the use of a mirror, marble, and linear measuring device



(Kidd, Myers, & Cilley, 1970, p. 248). For the sake of simplicity, assume the ground is horizontal and that the flagpole is perpendicular to the ground. The marble is used to assure that the mirror lies in the plane of the ground. If the marble rolls off one edge of the mirror, objects would be used to level the mirror before any measurements are made. An individual is positioned so that the top of the flagpole can be seen in the mirror. That mirror spot is marked mentally or physically and measurements are taken. The distance from the point on the mirror to a point below the person’s eye (AC) is measured. The height of the person’s eye above the ground (AB) can be determined. The distance from the point on the mirror to a point below the top of the object being used (CD) can also be found. The similar triangles are used to find the missing height. Triangle ABC is similar to triangle CED. Knowing that the ratio AB:AC is the same as CD:CE and being able to measure AB, AC, and CD permits calculation of DE.

Discussions about locus can provide interesting excursions into areas that will achieve attention-grabbing objectives for some students. Mainly, with the use of dynamic software, students can be enticed to conjecture what will happen as the locus of a point or line is investigated. For example, given two points, what is the locus of all points equidistant from them?



Segment CD is the radius of the circles centered at A and B. E and F are equidistant from A and B. Tracing the locus of E and F as CD changes quickly reveals that the desired result is the perpendicular bisector of segment AB. Students could be asked to develop the construction and conjecture the results before the actual trace. One difficulty with this approach is that the students will know the power of the software, and there is a temptation to use it to answer the question without giving the question much thought. The students who possess this intuitive “feel” for what the situation encompasses should be better equipped to deal with establishing an advanced-level proof.

Students have traditionally been required to construct perpendicular lines using a straightedge and compass. This has historical roots. It is a way to help students see and gain a “feel for” right angles. Another method of accomplishing the task of forming perpendicular line segments is to fold a piece of paper. Open the paper and then refold it so that part of the initial fold lies on top of itself. Opening reveals perpendicular line segments. Repeating the process of folding part of one segment onto itself at a different point will yield parallel line segments. The third segment, perpendicular to the other two, serves as a transversal. A fourth fold could be created to establish an oblique transversal. Folding with waxed paper is advantageous because its translucence permits easier multiple folds. Other classic folding activities involve bisecting an angle and constructing a parabola by folding a point onto a straight line segment. Patty Paper geometry (Serra, 1994) has come onto the scene as a distinct topic in the study of geometry.

In most instances, informal geometry is inserted into the curriculum as separate features in the textbook. As the trend to integrate topics grows, the likelihood is that there will be more informal geometry. There is one text on the market carrying the title Informal Geometry (Keedy, Bittinger, Smith, & Nelson, 1986). This text covers most of a formal geometry course, except rather than proving theorems, they are basically given as fact and the students work exercises using the theorems given. At first glance this may not seem like a good thing to do.

However, if the students gain insight and an intuitive feel for what the theorems project, that background can be used later when there is a need for more formal coverage of the topic. Of course, dynamic geometry programs could be used to generate this intuitive feel by manipulation of constructed figures.

GEOMETRY IN ALGEBRA

Peter Hilton stated that if you have algebra without geometry you have answers to questions nobody would ask, and if you have geometry without algebra you have questions you can't answer. If Hilton's statement is accepted, the separation of geometry and algebra is, perhaps, more tragic than one might think. If the two topics were treated together as needed, the curriculum would change. The amount of integration would increase. This flies in the face of tradition. Dare we do that?
Hero's (Hero of Alexandria, first century A.D.) formula for the area of a triangle A = $\sqrt{s(s - a)(s - b)(s - c)}$, where s = 0.5(a + b + c) can provide some interesting application opportunities. Many students are of the opinion that the only way to find the area of a triangle is to have the base and height. However, Hero's formula provides a method for finding the area of any triangle for which the lengths of the three sides are known. Suppose you are considering the purchase of a plot of ground that is 20 yards by 45 yards by 75 yards. The real estate agent is convinced the purchase price is very reasonable and is pushing for the sale. The property has not been surveyed, but that can be done once the nonrefundable deposit is submitted by you. Using Hero's formula,

$$
\begin{aligned}
s &= 0.5(20 + 45 + 75)\\
&= 0.5(140)\\
&= 70\\
A &= \sqrt{70(70 - 20)(70 - 45)(70 - 75)}\\
&= \sqrt{70(50)(25)(^{-}5)}\\
&= \sqrt{^{-}437500}
\end{aligned}
$$

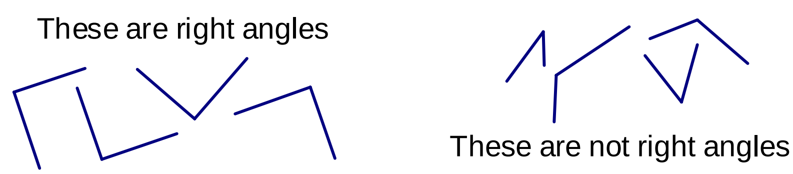
Something is wrong! Algebra provided an answer to a geometry problem. There is not a triangle.

ORDER OF COVERING TOPICS

As in most areas of the curriculum, geometry textbooks are similar. Certainly there are differences, but examination reveals a long list of items that are comparable. This has to be, because of our system of establishing objectives that need to be met as a guide to completing a given course. Topics may be treated differently and some emphasized more than others, but still, the similarities are detectable. As is the case in most subjects, the geometry textbook often dictates the order in which topics are presented. That may or may not be ideal.

Two books deserve special mention. The first, Geometry: A Guided Inquiry (Chakerian, Crabill, & Sherman, 1987), is available as a study, investigation, and learning tool. Activities are interspersed throughout the text and provide formative background preparing the student for more formal proofs. The thinking is that the intuitive feeling the student gathers from the investigations and activities will stimulate a higher level of understanding. Furthermore, as questions are raised during these research moments, students will begin to want to establish why things are true. In this way, the concept of and need for proof begins to emerge naturally.

The second book is Discovering Geometry, An Inductive Approach (Serra, 1997). Examples that show what something is and is not are provided as a means of developing definitions. That way the students are actively involved in the learning environment as they are asked to define terms. An example of the type of presentations made for definitions in this text would be



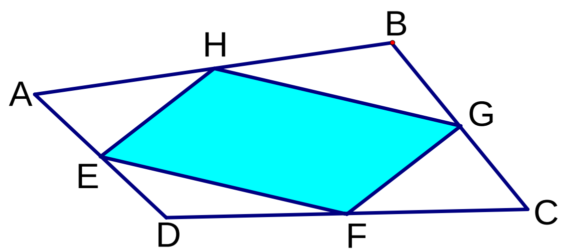
The major part of the Serra text has students doing a variety of activities and applications of geometry and there appears, at times, to be limited connection with either each other or with the ideas normally covered in a formal high school geometry course. However, the last part of the text asks the students to produce proofs. At this point, a wondrous thing happens - - the students have a wide variety of intuitive feels for what needs to be done in the proof. It is at this point that the realization begins to surface that these inspirations are a result of the groundwork laid in the beginning parts of the text. Amazingly, a wide variety of proofs is covered quickly and, most important, the attitude about them is generally quite positive.

One final note on the order of teaching geometric topics. Howard Eves stated that it seems reasonable to let the history of mathematics be our guide as to the order in which topics are introduced. His thinking is that the development is following a natural order generated by need and expansion of known items. Following Eves' line of thought, many of the developmental topics can be motivated early on. Then, as the curriculum becomes increasingly integrated, more extensive coverage can be offered as the students are developmentally ready. Until that time, we will need to rely on the formative work being done in more traditional time frames, but we can still look to the history of mathematics as a guide for order.

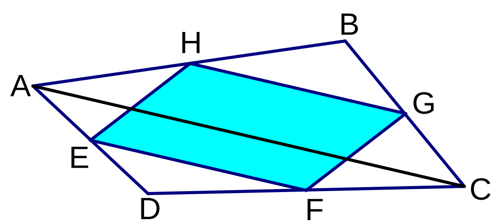
FORMAL GEOMETRY

Is there a need for a formal geometry course? Absolutely. The format of the course may have changed and there might be less emphasis on proof than there was in previous years. However, the value of the course must not be overlooked. Some question the wisdom of changing the course from one of proofs essentially every day all year long. Others see that change as a breath of fresh air, citing that students did not like proofs, the course turns many students off from mathematics and, developmentally, students are not ready for the rigor demanded in the course. Recall the earlier comments about the work of the van Hieles and the idea that the current geometry courses might be being watered down so our students can tolerate them.

Regardless of the philosophic position adopted, there is a place for the formal geometry class in the curriculum. Intuition is to be developed prior to the course and then a more formal approach used in the course itself. For example, how could we show that the interior quadrilateral EFGH is a parallelogram?



A proof is relatively simple if the students realize that each part of the original quadrilateral ABCD can be a triangle as indicated with drawing line segment AC.



When it is known that the midpoints of two sides of a triangle are joined by a segment that is half the length of, and parallel to, the third side, the establishment of a more formal proof that EFGH is a parallelogram is easier.

WHERE DOES GEOMETRY STOP?

Geometry study and learning do not stop. Geometry is a dynamic, ever-changing subject. Transformations. Tessellations. Vectors. Coordinate. Taxicab. Lobachevskian. Spherical. Flatland (Abbot, 1963). One, Two, Three . . . Infinity (Gamow, 1961). Donald Duck in Mathmagic Land (Disney, 1959). Measurement. Projection. And so on.

Tessellations can be entertaining and thought provoking. The Dutch graphic artist Maurits Corneille Escher [esh'-ur] has been one agent in the popularizing of tessellations. "His work has become increasingly popular because of its unique combination of humor, logic, and meticulous precision with visual trickery" (Escher, New Grolier Multimedia Encyclopedia, 1993). A square tessellates the plane, meaning that a set of squares of a given size can be arranged to cover the plane leaving no gaps. Escher and many others modify a shape like the square to present more interesting creations.

The honeycomb is a practical application of the tessellation of the plane. It is important to show students applications as well as connections within geometry. The study of geometry does not stop. We live in a geometric world. As we view surroundings, new questions are raised and the need for varied geometric understandings continues.

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