Knowing where to begin this workshop and what to include is difficult. There are so many topics to be covered. What is the best order? What background information will the students possess? Should topics be integrated in this discussion? You are expected to blend topics as you teach by relating one to the other. Following an integrated format makes continuity next to impossible. This comment brings to mind the story about an individual wanting to borrow a chainsaw. The request was made and the response was “No.” Rather stunned, the individual followed with “Why not?” “My dog is sick,” was the reply. “What does your dog being sick have to do with my borrowing your chainsaw?” was the next question. The response was “Nothing, but I don’t want to lend it to you and one excuse is as good as another.” The moral of that story is that no matter what order is selected, others could serve equally well.

THE STANDARDS

The NCTM Standards (1989 & 2000) provide a listing of basic concepts and objectives that should be covered in. The discussion in the Standards includes rationalization for shifting to a broader curriculum of mathematics, features that should be included in the curriculum, commentaries on technology, comments dealing with instruction, materials that should be present in every classroom, and learner characteristics. The bottom line is that the Standards provide a solid guideline for what mathematics needs to be taught in all programs.

INTEGRATING TOPICS WITHIN MATHEMATICS

As a teacher of mathematics, you are consistently asked to show applications of the material being covered. There is an increased emphasis on blending mathematics with the other subjects being taught. Science is rather easy to do this with because of the many formulas used there. Geography can be done with relative simplicity as global locations are discussed. How about English?

Mark Twain in Life on the Mississippi (Adler, 1972, Book 2, pp. 56–58.) described a series of events involving the Mississippi River abandoning a course and acquiring a new, shorter one when considering the distance traveled between Cairo, Illinois, and New Orleans, Louisiana:

> The water cuts the alluvial banks of the “lower” river into deep horseshoe curves; so deep, indeed, that in some places if you were to get ashore at one extremity of the horseshoe and walk across the neck, half or three-quarters of a mile, you could sit down and rest a couple of hours while your steamer was coming around the long elbow at a speed of ten miles an hour to take you on board again. When the river is rising fast, some scoundrel whose plantation is back in the country, and therefore of inferior value, has only to watch his chance, cut a little gutter across the narrow neck of land some dark night, and turn the water into it, and in a wonderfully short time a miracle has happened: to wit, the whole Mississippi has taken possession of that little ditch, and placed the countryman’s plantation on its bank (quadrupling its value), and that other party’s formerly valuable plantation finds itself away out yonder on a big

> island; the old watercourse around it will soon shoal up, boats cannot approach within ten miles of it, and down goes its value to a fourth of its former worth. Watches are kept on those narrow necks at needful times, and if a man happens to be caught cutting a ditch across them, the chances are all against his ever having another opportunity to cut a ditch. . . . Since my own day on the Mississippi, cutoffs have been made at Hurricane Island, at Island 100, at Napoleon, Ark., at Walnut Bend, and at Council Bend. These shortened the river, in the aggregate, sixty-seven miles. In my own time a cut-off was made at American Bend, which shortened the river ten miles or more.
>
> Therefore the Mississippi between Cairo and New Orleans was twelve hundred and fifteen miles long one hundred and seventy-six years ago. It was eleven hundred and eighty after the cut-off of 1722. It was one thousand and forty after the American Bend cut-off. It has lost sixty-seven miles since. Consequently, its length is only nine hundred and seventy-three miles at present.
>
> Now, if I wanted to be one of those ponderous scientific people, and “let on” to prove what had occurred in the remote past, by what had occurred in a given time in the recent past, or what will occur in the far future by what has occurred in late years, what an opportunity is here! Geology never had such a chance, nor such exact data to argue from! Nor “development of species,” either! Glacial epochs are great things, but they are vague—vague. Please observe: In the space of one hundred and seventy-six years the Lower Mississippi has shortened itself two hundred and forty-two miles. That is an average of a trifle over one mile and a third per year. Therefore, any calm person, who is not blind or idiotic, can see that in the Old Oolitic Silurian Period, just a million years ago next November, the Lower Mississippi River was upward of one million three hundred thousand miles long, and stuck out over the Gulf of Mexico like a fishing-rod. And, by the same token any person can see that seven hundred and forty-two years from now the lower Mississippi will be only a mile and three-quarters long, and Cairo and New Orleans will have joined their streets together, and be plodding comfortably along under a single mayor and a mutual board of aldermen. There is something fascinating about science. One gets such wholesale returns of conjecture out of such a trifling investment of fact.

There are other wonderful connections between mathematics and literature. Flatland (Abbot, 1963) is a discussion about a two dimensional world and the many strange things that happen there. “T. C. Mits” (The Celebrated Man in the Street) is a poetic rendition of a multitude of topics starting with mathematical basics and extending through and beyond calculus in which application is discussed.

Why is it that connections between mathematics and other curricular areas are often not made or, if they are, it is only at some superficial level? Part of the answer is in the idea that the mathematics you learned and teach has always been pretty much self-contained. Each topic was treated at the necessary level and then the curriculum (or text) moved on to something else. At times, yesterday’s topic was connected to today’s, like prime and composite numbers being followed by GCF (greatest common factor) and LCM (least common multiple).

Students frequently have trouble finding the GCF and LCM and deciding which one they have after they have “shoved the numbers around.” Here is a method that it is very easy to follow and remember, which helps find GCFs and LCDs. Suppose the problem is to find the LCM and GCF of 16 and 24.

What goes into both 16 and 24? $2\ \underline{|16\ \ 24}$

Two is a common factor of both 16 and 24. Other values could be selected, but 2 is a typical reflexive answer of students. They could say 8, or even 4, but often they do not see that (particularly weaker students) and thus, the typical response of 2. $\begin{array}{r|cc} 2 & 16 & 24 \\ \hline & 8 & 12 \end{array}$

You now are left with 8 and 12 from the original numbers and the process is repeated. What goes into both 8 and 12? $4\ \underline{|8\ \ 12}$, leaving $\begin{array}{r|cc} 4 & 8 & 12 \\ \hline & 2 & 3 \end{array}$.

Two could have been selected again, instead of 4. Regardless, eventually the final pair of numbers will be relatively prime, which is the case with 2 and 3 in this example.

Two and 4 are factors of the GCF (8), and the factors of the LCM (48) are 2, 4, 2, and 3. This is much easier to see if the whole presentation is consolidated:

$$\begin{array}{r|cc} 2 & 16 & 24 \\ \hline 4 & 8 & 12 \\ \hline & 2 & 3 \end{array}$$

Notice how the 2, 4, 2, and 3 form an “L”, indicating the factors of the least common multiple. Notice also that the factors in the vertical part of the “L” comprise the GCF of 16 and 24 (you are not limited to two factors either vertically or horizontally in the “L”. Notice how this process makes a connection between GCF and LCM.

There is a message here - - connection of topics is possible. More than likely the above process is new for you, which generates another message - - the way you learned to do things may not be the best one for your students. That in turn brings up another message - - it is imperative that you be a lifelong learner in the field of teaching mathematics. How do you accomplish that task? Read professional literature. Attend professional conferences that focus on the teaching of mathematics. Participate in in-service opportunities. Enroll in additional classes (both pedagogy and content). Watch and listen to your students learn. They will give you all sorts of clues about how to become a more effective teacher of mathematics. Finally, think about what you are doing. Please do not reflexively follow the book or do it the way you have always done it. Think about connections between topics and subjects.

Regretfully topics often are not connected. How many times have you seen LCM related to LCD (least common denominator)? As fractions are added, how often are equivalent fractions mentioned? Have you ever wondered why equivalent fractions, which to some extent employ multiplication of fractions, are discussed before multiplication of fractions? The list goes on. You are asked to integrate topics, and yet the presentations given to you for reference to work from are not done in that manner. Is it fair to expect you to do that?

Why is topic integration so uncommon? Perhaps the writers and teachers think that students will not make the desired connections. Maybe the teachers never thought of making those bridges between topics. After all, today's classroom teachers had little opportunity to see this style of teaching in their school experiences. There could be a lack of exposure to areas in life where experiences provide the necessary background from which to build the requested associations. Look at the other individuals who are studying to become teachers of mathematics. The person who graduates from high school, goes directly to college, and then into teaching is not uncommon. The newly graduated teacher of mathematics could well be a 21- or 22-year-old who possesses volumes of book learning with limited life experience. How can that individual be expected to relate the learning of mathematics to a variety of world applications? In that context, how practical does the request to integrate subject areas seem?

Integration of topics, although most desirable both within mathematics and between subjects, is not an easy task. It is going to require reflective thought. You will need to consider a topic and all the connection points for it, in the mathematics classroom, the school context, and the world in which the students live. This has to be the world as the students see it, not the world that we as adults think the students see. There is a difference! These thoughts will begin to reveal paths to follow that can make the subject you discuss more real, dynamic, and interesting to the students. That is your job as a teacher of mathematics!

THE BIG 20

The Big 20 was developed by an individual with many years of experience teaching high school students who were not particularly motivated to learn, let alone to learn mathematics (personal conversation with G. Rule, Chuluota, FL, 2000). Initially the collection consisted of about 50 problems, but items were gradually eliminated (duplication or not essential for daily survival). It now consists of 20 problems that use multiplying and dividing by powers of 10; adding, subtracting, multiplying, and dividing whole numbers, fractions, and decimals; percent; money; converting decimals to fractions or percent (and all other combinations of these three); and finding squares and square roots of numbers. The contention is that if the skills necessary to complete The Big 20 are mastered, students will have command of the basic ideas necessary for mathematical survival in daily life. The skills are ones encountered regularly, either directly as they are shown here, or as parts of other problems.

When the idea of the Big 20 was first introduced, it seemed quite simple to students and teachers of “higher” classes. Claims that students in the “other” mathematics classes could do the Big 20 in less than 2 minutes (without calculators) did not seem too preposterous. The Big 20 has been given to several college classes and groups of teachers. Mr. Rule required his “lower” mathematics classes to complete the set of problems in no more than 2 minutes, missing none. Try it. No one has been able to meet the challenge when The Big 20 has been presented to teachers or college students the first time. With practice, it can be done. Mr. Rule’s students could all do the Big 20 in 2 minutes or less, missing no more than one question by the Thanksgiving break each year. Additional attempts would find different problems, perhaps in an alternative order; but they would be similar to the ones shown here.

1. \$57 divided by 10 = ___________
2. \$627 divided by 100 = ___________
3. \$48 divided by 1000 = ___________
4. \$8.45 x 10 = ___________
5. \$0.28 x 100 = ___________
6. \$12.24 x 1000 = ___________
7. \$0.05 x \$0.12 = ___________
8. \$18 - \$0.22 = ___________
9. \$15 - \$0.15 = ___________
10. 0.82 as a fraction = ___________
11. $(5/8)^2$ = ___________
12. \$52.6249 = ___________
13. 3 $^1/_2$ - 2 $^1/_4$ = ___________
14. 5 $^1/_4$ - 3 $^1/_2$ = ___________
15. 7 divided by 1 $^1/_4$ = ___________
16. $^1/_4$ x 2 $^1/_2$ = ___________
17. 5 $^1/_2$% of \$60 = ___________
18. 8.65 divided by 0.05 = ___________
19. $^1/_4$ as a decimal = ___________
20. $(0.2)^2$ = ___________

START

It is your job to stimulate your students to want to learn the mathematics placed before them. You must create in them the wisdom that these topics are needed. They have got to understand that these concepts establish a foundation on which they will build for the rest of their academic, work, and recreational lives. It is difficult to do this because in many instances, the topics being covered have been dealt with earlier in the curriculum. Why then, are they being treated again? There are two simplistic and yet very realistic reasons:

Tradition

Students do not know the material well enough to go on.

Tradition is difficult to change. It is almost like paying one's dues - - I had to go through it more than once to assure I had mastered it and so do you. Remember the chainsaw story?

Lack of knowledge about a particular topic is a more complex issue. How do we justify treating the same concept year after year? One reason is the spiral approach embedded in our curriculum. A topic is discussed to the depths students can handle it and then abandoned until a later date when additional background information and readiness have been established. At that time, the topic is treated in greater detail or depth, expanding the horizons of the students in a manner where they see that the new information has built on the prior experiences. The problem is, even though this sounds like a reasonable approach, often the coverage is the same as before. Why? Perhaps some of the students in the class have not mastered the background material well enough to permit advancement. What about those who have? Investigation of subsequent treatment of topics often reveals that the same tunes and words are used by teachers; only the voice, pace, and inflections change. The rest of this workshop deals with topics visited more than once in our mathematics curriculum.

FRACTIONS

Formative concepts relating to fractions are introduced in the early school years. They are built on and expanded throughout the curriculum. More difficult fractions are encountered with rational expressions in algebra. The procedures are the same, but the complexities involved, because of the polynomials and associated necessary skills, are much greater. Still, topics like rational expressions are dependent on the basic skills of working with fractions. Consider addition of two fractions with denominators that are relatively prime. Students are told:

Find the LCD (least common denominator).

Divide the denominator of the first fraction into the LCD and multiply the numerator of that first fraction by the determined missing factor from the division.

Divide the denominator of the second fraction into the LCD and multiply the numerator of that second fraction by the determined missing factor from that division.

Add the two products in the numerators.

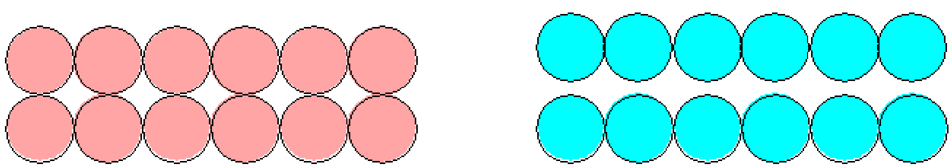
Put the answer over the LCD.

Students encounter this topic several times. Certainly the words might be different, but those alterations are only to meet local or personal preferences. The ideas are still the same. Students say, “I have heard this before. I know what to do,” and they tune you out. Perhaps the students are correct as they say they know what to do. If so, why is the topic covered again? Then again, maybe the students do not know what to do, but they think they do. Often, this provides an opportunity for creative mathematics where students take bits and pieces of explanations and combine them to create some completed whole, disregarding whether or not the development is correct. The concern of many students is to do something so you will not focus on them or their behavior. The assumption seems to be, “As long as I act busy, I am safe.” There is no concern for correct responses. Students are satisfied to have done something and gotten you “out of their face” for a while.

Student attitudes as described at the end of the last paragraph are devastating, and yet they are all too common. How can that mentality be combated? How does a teacher go about getting students to want to learn material, whatever it is? One idea is to sing a different tune and use different words. While this may seem very childish, the addition and subtraction of fractions could be covered with a carefully prepared sequence of problems and egg cartons. Most of the sequence will be shown to indicate what can be done.

Each student should have a set of cartons. They build should build their set to help them understand the pieces. Remove the lids and tabs from dozen cartons (while other sizes can be used, consistency helps with the explanations that follow). Use different colors for each part, if possible. Select one full dozen bottom (2x6) to be the unit (pink for our discussion).

Cut the second bottom (blue) into 2 EQUAL PARTS (making 1x6 rectangles)

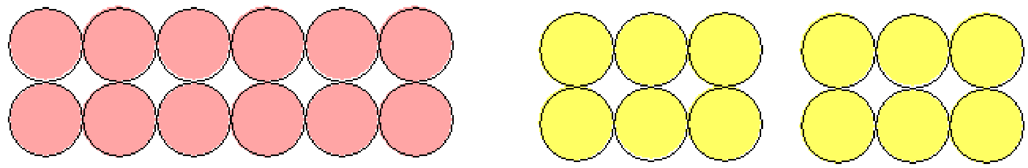


Verify each part is $\frac{1}{2}$ by saying and showing that it is ONE OF 2 EQUAL PARTS.

EQUAL PARTS MUST be STRESSED throughout this approach.

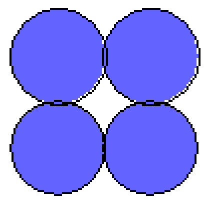
Comment when pink is the unit (or one), blue is one of 2 = parts or $\frac{1}{2}$ of that unit.

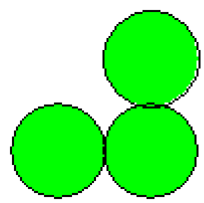
Cut a third bottom (yellow) into 2 = parts (2x3 rectangle)

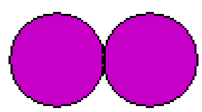


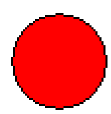
Verify that each part is $\frac{1}{2}$ by saying and showing that it is ONE OF 2 EQUAL PARTS.

Cut additional bottoms for each of the following verifying each part and saying and showing that it is ONE OF so many EQUAL PARTS:

$\frac{1}{3}$ making three 2x2 squares like 

$\frac{1}{4}$ making four "L" shaped figures like 

$\frac{1}{6}$ making six 1x2 rectangles 

$\frac{1}{12}$ making twelve 1x1 rectangles (just 12 holes) 

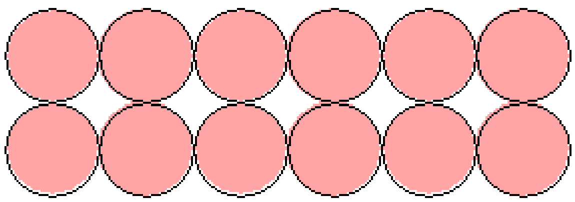
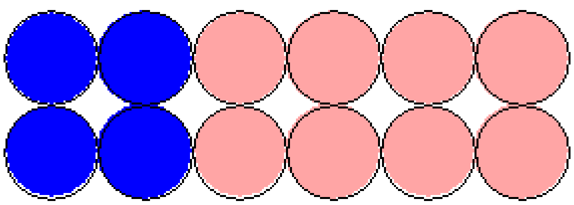
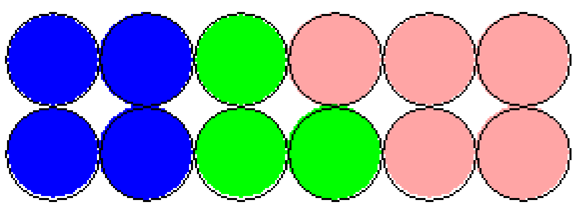
Make sure students recognize different parts when PINK IS UNIT because the expression is important later when unit dimensions change.

Do equivalent fractions with the complete bottom (2x6) as unit.

A 2 by 2 set represents $\frac{1}{3}$.
It takes two 1 by 2 sets to fill a 2 by 2 piece.
That means two 1 by 2s are the same as one 2 by 2 piece.
But, a 1 by 1 piece is $\frac{1}{6}$ when pink is the unit, and a 2 by 2 represents $\frac{1}{3}$.
So, two $\frac{1}{6}$s are the same as one $\frac{1}{3}$.

It must be the case that $\frac{2}{6} = \frac{1}{3}$, showing equivalent fractions.

Do similar activities using different sized bottoms, getting students accustomed to the idea that any sized carton can serve as the unit. Suppose a 2 by 3 rectangle is the unit. Then it must be the case that each hole represents $\frac{1}{6}$ of the unit. A rectangle that is 1 by 2 represents $\frac{1}{3}$ of the unit at a 1 by 3 "L" configuration represents $\frac{1}{2}$ of the unit. It is important to do a lot of these before proceeding to addition of fractions with egg cartons.

When adding we put things together. We will start with unit fractions only, and use relatively prime denominators. Consider $\frac{1}{3} + \frac{1}{4}$ using a 2 by 6 bottom as the unit.  Put $\frac{1}{3}$ inside the unit at one end  and put a $\frac{1}{4}$ next to the $\frac{1}{3}$, also inside the unit  showing $\frac{1}{3} + \frac{1}{4}$. How many holes of the unit are filled? 7. How many possible holes are there in the unit? 12. So, 7 out of the 12 holes, or $\frac{7}{12}$, is filled, showing that $\frac{1}{3} + \frac{1}{4} = \frac{7}{12}$. Notice there was no discussion about LCD and yet the sum is correct. The point is that the fractions have been added correctly without any of the finding the LCD, dividing denominators into that LCD, etc. This is very helpful to a student who is struggling with all of the details.

It is desirable to use the smallest possible unit when using egg cartons. This avoids the need to divide out common factors later (see reducing fractions in the

Skills Workshop). Finding the smallest unit is not difficult and this section will help them realize that, along with the idea that the size of the unit can be changed.

Suppose the task is to find the unit needed to find $\frac{1}{2} + \frac{1}{3}$ (our assumption is that they do not understand finding the LCD because if they do, this whole process is not necessary - - UNDERSTAND is the operative word here).

Can 1 hole be unit?
- No because you cannot get half of it (a hole is the smallest piece)

Can 2 holes be unit?
- You can get a half.
- But, you cannot get a third.
- Both a half and a third are needed so this cannot be the unit.

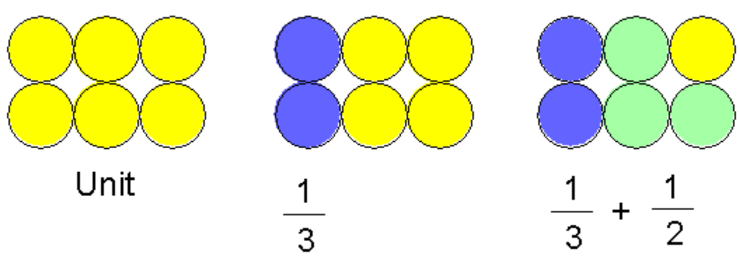
Can 3 holes be the unit?
- You can get a third.
- But, you cannot get a half.
- Both a half and a third are needed so this cannot be the unit.

Can 4 holes be unit?
- You can get a half.
- But, you cannot get a third.
- Both a half and a third are needed so this cannot be the unit.

Can 5 holes be unit?
- But, you cannot get a half or a third.
- So this cannot be the unit.

Can 6 holes be unit?
- You can get a half.
- You can get a third.
- This is the first set that gives both a half and a third.
- This is the unit.

We are now able to do $\frac{1}{2} + \frac{1}{3}$ with the cartons:



It is time to begin recording the results so far. We have shown

$\frac{1}{3} + \frac{1}{4} = \frac{7}{12}$,

$\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$, and we could show more, like

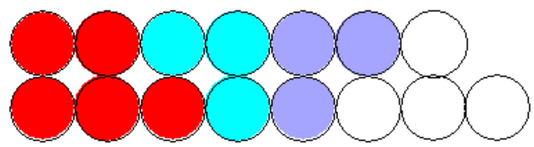
$\frac{1}{3} + \frac{1}{5} = \frac{8}{15}$,

$\frac{1}{4} + \frac{1}{5} = \frac{9}{20}$, and so on. The number that needs to be done here will vary with the students. Hopefully some of them will start observing and generalizing things like:

The answer denominator is the product of the 2 problem denominators
(This becomes the LCD later)
The answer numerator is the sum of the 2 problem denominators.

While this is not quite correct for a total generalization, good groundwork is laid.

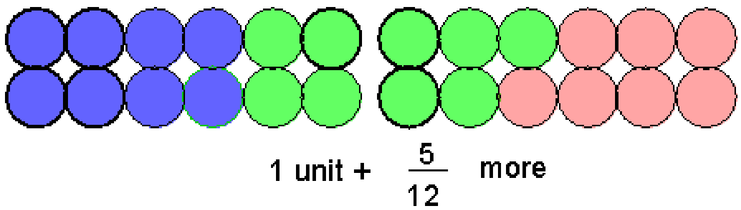
Next consider non-unit fractions. When this part is done, a slight alteration to the above generalization will yield the rule commonly used for adding fractions with different denominators. Suppose the problem is $\frac{1}{3} + \frac{2}{5}$. The egg cartons can be used to show the sum is $\frac{11}{15}$.



$\frac{1}{3} \quad + \quad \frac{2}{5}$

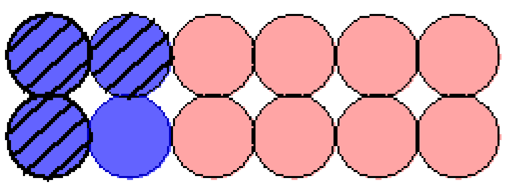
We know we have two of the fifths. We also know the sum is $\frac{11}{15}$. The denominator part of our generalization still works! But adding the problem denominators only gives a sum of 8 and a sum of 11 is needed. Is there any combination of 3s and 5s that will give a sum of 11? Two 5s and a 3 will not work, but two 3s and a 5 add to 11. Essentially we have cross multiplied the numerator of one fraction (2) and the denominator of the other fraction (3) to get 6, which when added to the 5, gives the desired sum of 11. Careful examination shows that the cross multiplication happens with the 5 and 1 too, and that it was happening in the initial examples as well. It is just that since one is the multiplicative identity, the cross multiplication is more difficult to notice.

Extending the discussion to $\frac{2}{3} + \frac{3}{4}$ will amplify the cross multiplication idea. A 2 by 6 carton is the unit. The two thirds and the three fourths extend beyond the unit, so you have one full unit and some more. How much more? 5 holes more. A hole is worth $\frac{1}{12}$ so $\frac{2}{3} + \frac{3}{4}$ yields one full unit and $\frac{5}{12}$ or $1\frac{5}{12}$, which also happens to be $\frac{17}{12}$. But 17 = 8 + 9. That is, 17 = (2)(4) + (3)(3), and the cross multiplication idea is confirmed.



Certainly there would need to be some discussion about conversion between improper fractions and mixed numbers. The goal is to abandon the egg cartons as soon as reasonable possible, but do not rush away. Let the students begin to generalize and eventually conclude that the entire operation of adding fractions with unlike denominators and converting between improper fractions and mixed numbers can be done mentally. Then, leave the egg cartons (with the caveat that they can be brought out again if needed).

Subtraction with egg cartons is similar to addition process. Suppose the problem is $\frac{1}{3}$ - $\frac{1}{4}$. The unit can be determined as a 2 by 6 egg carton. Put the $\frac{1}{3}$ in one end of the unit and then put the $\frac{1}{4}$ inside the $\frac{1}{3}$. How many holes of the $\frac{1}{3}$ are not covered by the $\frac{1}{4}$? In terms of the unit, what is that hole worth? $\frac{1}{12}$. So, it must be the case that $\frac{1}{3} - \frac{1}{4} = \frac{1}{12}$, as shown by



DECIMALS

In today's technologically based world, why isn't there more emphasis on decimals and less on fractions? It seems as if decimals appear more consistently in real-world settings. Most calculators starting with the basic scientific models and up operate with fractions (Casio fx 55, Casio fx 65, Casio CFX-9850GB Plus, TI 30X, TI Math Explorer, TI graphing calculators, etc.), but all calculators operate conveniently with decimals. Calculators are a part of the world in which the students live. One argument against permitting calculators in the school is that of expense. In an age where many students have more than one video game cartridge, cars, private phone lines, cellular phones, beepers, and part-time jobs, coupled with the low cost of basic calculators, it seems incomprehensible that price would be raised as a excuse for disallowing calculators. Of course, another excuse for not permitting the use of calculators is the age-old idea that I had to learn to cipher without a calculator and so should my students. Remember the chainsaw story? Then there is the line about tests not permitting calculators, which seems pretty weak when one considers the AP exam calculator expectations. Sounds like another chainsaw to me. ☺

Before going on, please realize that the preceding calculator statements are not to be construed as saying that students should not be required to drill and memorize basic facts, learn how to do the arithmetic algorithms, by hand, and so on. However, once a student has demonstrated the ability to apply the addition algorithm for whole numbers, for example, why not permit the calculator? Somewhere between the addition facts and something like 2345678 + 1357902 + 4268905 + 3265980 + 9876543 is a reasonable cutoff where a calculator should be used. That problem is a marathon, loaded with opportunity for errors, and does not amplify one's understanding of the basic addition algorithm.

If calculators would be permitted, how would the decimal curricular offering be altered? Certain readiness skills must be present before students can consider operating with decimals. These will be needed whether or not a calculator is available. If calculators are present, computation with decimals might be within the grasp of most students. Individuals opposed to the use of calculators rationalize that students should understand a concept prior to using the calculator. This could be used as an argument against the use of calculators early in the curriculum. We do not argue the value and necessity of understanding concepts. We totally agree. However, we must exercise caution that our zeal for "understanding a concept" is not an excuse to prevent the use of technology and teach as we always have. Remember, any excuse will do.

Calculator proponents contend that students have the opportunity to discover decimal operational procedures while they experiment with a calculator before formal instruction dealing with decimals. It should be noted that it is most desirable for students to have a good idea of what is going on as opposed to rote memorization of procedures and rules. For some reason, a vast majority of our population seem much more ready to embrace memorizing rules with no understanding of the concepts involved than they are with using technology where there is limited understanding of the concepts involved. Why is that? Minimal readiness skills for operation with decimals include place value; ability to multiply by counting number powers of 10; ability to divide by counting number powers of 10; realization that 3.2 = 3.20 = 3.200 . . . ; knowledge that decimals are fractions with denominators that are counting number powers of 10; and facts for all basic operations.

Close to the turn of the century, some state legislatures adopted the position that no calculators should be permitted in any school below seventh grade. The basic argument is that students need to know how to compute with the standard algorithms by hand. There is one huge flaw in this approach. It seems as if there is an assumption that if a student can perform an operation by hand, that student understands the workings of the algorithm. Nothing is further from the truth. Many students memorize how to do an algorithm and have absolutely no idea what is behind its function.

As a professional, you should visit the Mathematically Correct site (http://www.mathematicallycorrect.com). This is a part of helping you make an informed decision. If you are against new math in schools, this site presents plenty of fuel for your fire. It gives several articles on “the invasion of new math” in our schools. Plenty of enthusiasm is conveyed for their cause. There are articles on what is going on in teaching mathematics in the nation. Many texts are reviewed in detail on this site.

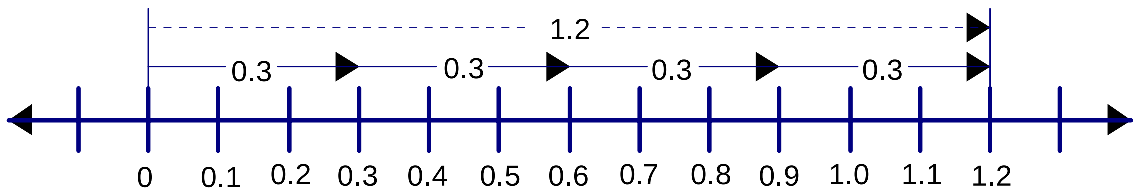
You should also visit Mathematically Sane, which was created in response to Mathematically Correct, and pretty much adopts an opposing position. Their address is http://www.mathematicallysane.com/home.asp.

Two potential problem areas exist with addition of decimals: regrouping and ragged decimals. Finding the sum of 3.4 + 5.27 is an example of ragged decimal addition. If the student uses a manipulative like the base 10 blocks, interpretation is easy, as is the case with pictures or letters representing the blocks. The difficulty arises at the abstract level. One readiness skill for students is the realization that 3.4 = 3.40. If that is in the students’ background, 3.4 + 5.27 poses no particular obstacle since it can be rewritten as 3.40 + 5.27 and the ragged feature is eliminated. Soon, most students should realize that the writing of the zero is not necessary. If that skill is not present, it is reasonable to ask why the student is being asked to perform this task without appropriate background capabilities.

The answer to regrouping situations lies in readiness founded in the base 10 blocks. Students learn early in block use that ten ones are equivalent to one ten and the substitution can be made at any time—either way: one ten for ten ones, or ten ones for one ten. Extending this background to include ideas like one tenth is equivalent to ten hundredths, and most problem areas typically associated with regrouping in decimal addition (and subtraction) are eliminated.

The problem 9 – 2.7 can be done the same way 800 – 372 was discussed in the Skills in Teaching Mathematics Workshop. The 9 can be expressed as 8.9 + 0.1. The subtraction can now be viewed as 8.9 - 2.7, which is 7.2. However, that 0.1 needs to be added back in to keep the numbers in balance so the answer is 7.3. You should note that this example has some duplication of digits. Duplication can be confusing and initially should be avoided for the sake of student understanding because students often track numbers as they learn a new procedure. With duplicated digits, they sometimes lose track of which digit went or came from where, lessening the likelihood that they will understand the process quickly.

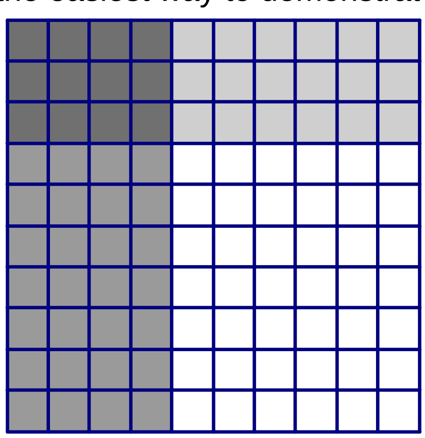
Before doing decimal multiplication, it must be assumed that the students know their multiplication facts and have a basic grasp of the concept of multiplication in general. Without these readiness skills, one must ask why a student would be expected to multiply decimals. A carefully devised sequence is extremely helpful to assist students in the learning of multiplication of decimals—something that is not always attended to in texts or established objectives. The first problem type would be a whole number times a tenth (4 × 0.3), which can be modeled on a number line as repeated addition.



After several similar problems, the students should conclude that this is like multiplying 4 and 3 except there is a decimal point in the answer. At this stage, with problems yielding a two-digit product, it is not known how the decimal point is located in the product. It is advisable to avoid something like 5 × 0.4 because the product of 2.0 looks unusual and might generate confusion that is not needed. Eventually though, 5 × 0.4 must be dealt with.

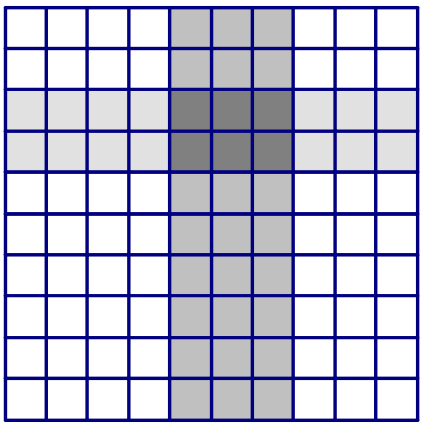
Problems like 4 × 0.3 should be followed by another series such as 4 × 0.03. This product can be shown on the number line like the one used for 4 × 0.3. The results should be similar to the generalization derived earlier except that the answer will now be 0.12. If you think this type will be a problem for students, you could give them problems like 4 × 0.21 to be developed on the number line. Regrouping should be avoided for the time being. The number of eligible examples is limited, but enough can be generated to give students the idea. At this point, the foundations of the rule for locating the decimal point in the product have been laid, although only one factor has had a decimal in it.

The next stage in the sequence involves a problem like 0.3 × 0.4, and arrays are the easiest way to demonstrate the product.



The horizontal band represents 0.3 of the unit square, and the vertical bar represents 0.4 of the unit square. The darker rectangle at the top left corner of the unit square represents the process of finding 0.4 of 0.3, or 0.4 × 0.3. The unit square is subdivided into 100 smaller congruent squares, each of which represents 0.01 of the unit square. Thus, the rectangle showing 0.4 × 0.3 occupies 12 of the smaller squares and is expressed as 0.12, the desired product.

The final problem in the sequence is similar to 0.3 × 0.2. This type poses an extra degree of difficulty because of the zero that must be inserted between the decimal point and the 6. This is contrary to what students have been taught previously and thus takes care to develop. An array model can be particularly useful here too.



The horizontal band represents 0.2 of the unit square, and the vertical bar represents 0.3 of the unit square. The darker rectangle where the bands overlap in the unit square represents the process of finding 0.2 of 0.3, or 0.2 × 0.3. The unit square is subdivided into 100 smaller congruent squares, each of which represents 0.01 of the unit square. Thus, the rectangle showing 0.2 × 0.3 occupies 6 of the smaller squares and is expressed as 0.06, the desired product. If the smaller squares are counted, the inserted zero to show 0.06 seems a bit more natural.

At this stage, the students should be ready to produce a generalization about the location of a decimal point in a product that is an approach to the rule traditionally given to classes. The difference is that the concrete approach and careful sequence of problems has led the students to the conclusion, as opposed to just telling them.

The only remaining decimal operation is division. All the difficulties and problems associated with division of whole numbers are still present when considering decimals. Decimals merely magnify all the trouble spots. Effective employment of the division algorithm demands that students:

Know place value
Be able to round
Have mastered estimation
Understand where to put a digit in the proposed answer
Have the ability to multiply when the two factors are not together and place the product in a location away from either factor

Subtract where regrouping is involved
Divide when the product is in one place, the factor is in another and the missing factor part is to be located in a third
Repeat the above cycle as needed
Locate the decimal point in the answer.

Locating the decimal point becomes the major trouble area. There are four possible division situations:

Whole number divided by a Whole number $\left(\frac{W}{W}\right)$

Decimal divided by a Whole number $\left(\frac{D}{W}\right)$

Whole number divided by a Decimal $\left(\frac{W}{D}\right)$

Decimal divided by a Decimal $\left(\frac{D}{D}\right)$.

While divisor, dividend, and quotient are common words for the different numbers in a division problem, they are not desirable. By using factor, product, and missing factor, the division process is connected to multiplication. In the long run, this connection enhances understanding.

The first two, where the factor (divisor) is a whole number, present the least amount of difficulty for students. The decimal location is a matter of placing it directly above where it appears in the product (dividend). When the factor is a decimal, there is a need for a change - - to make the factor a whole number. Traditionally the students have been told to move the decimal to make the factor a whole number and then move the product's decimal the same number of places and in the same direction. This rule seems obvious and clear to us, but we have seen it for several years. Do we understand why it works, or did we, at some time in our past, just accept it? Consider 9.6 ÷ 2.4, where the factor can be made a whole number by multiplying it by 10 and 2.4 becomes 24. At the same time, 9.6 must be multiplied by 10, giving 96, because if the denominator of a fraction is multiplied by a value, the numerator of that fraction must be multiplied by the same value. This process yields an equivalent fraction. The factor is a whole number and can be dealt with. This procedure gives an explanation of why we tell students to move the decimal point to the right as we divide decimals. This insight into why things are done enhances the possibility for a student understanding what is going on. Regretfully, many students end up memorizing the process and have no idea why they do what they do.

The coverage of decimals is not complete but the groundwork has been laid. Having done that, a fundamental question deserves consideration. Why do we spend so much time teaching students to deal with decimals when calculators are available and deal with the situations so readily?

One decimal setting appears impossible to students. Even when they see a "proof" that $0.\overline{999} = 1$, they resist accepting it. The typical development would include a discussion that $0.\overline{999}$ is a repeating decimal that never ends. It would be distinguished from 0.9, 0.99, 0.999, and so on, each of which terminates and, thus, can be expressed as $\frac{9}{10}$, $\frac{99}{100}$, $\frac{999}{1000}$, and so on, respectively. Assuming the students possess some algebraic background and are able to subtract one equation from another, consider the following approach to why $0.\overline{999} = 1$.

Let $x = 0.\overline{999}$		(1) Given
$10x = 9.\overline{999}$		(2) Multiply (1) by 10
Subtract (1) from (2)		
	$10x = 9.\overline{999}$	
	$-\ x = 0.\overline{999}$	(3) Subtracting (1) from (2)
	$9x = 9.000$	(4) Collect like terms
	$x = 1$	(5) Divide (4) by 9
But,	$x = 0.\overline{999}$	Given
So,	$1 = 0.\overline{999}$	Transitive property of =

Assure students that the 9s repeat in $0.\overline{999}$ and in $9.\overline{999}$ as far as they want, and usually they will accept the subtracting of 9 from 9 in any place value to the right of the decimal, no matter how far to the right it is. This brings them to the conclusion that the missing addend for $9.\overline{999} - 0.\overline{999}$ is 9, which is generally accepted with little resistance. Only when the division is completed in Equation (5) and it is concluded that $1 = 0.\overline{999}$ does the resistance begin to build. You need to be extremely convincing in your discussion to get many students beyond this resistance. Then, you can tackle more complex repeating decimals as fractions situations using a similar approach. As repeating decimal values are converted to fractions, a multitude of skills is practiced. With proper emphasis on your part, the students can begin to see the value of having a command of some of the skills we ask them to learn.

AMUSEMENTS, FASCINATIONS, AND BEYOND

For a multitude of reasons, students are not excited about the learning of mathematics. That is tragic, particularly in light of the idea that most children enter elementary school liking and anticipating learning mathematics. Still, for most of them, when they leave elementary school, their attitudes and outlooks have changed to that of rejection. Why is it that so many students view mathematics as boring, having no useful value in their world, and of little worth as they consider career options?

The teacher, along with excitement, enthusiasm, sincerity, an interest in students, adequate background knowledge, and stellar teaching skills, can have a tremendous positive impact on student learning. Assuming that, where does one begin to entice students back to being interested in learning mathematics? Games and tricks are a beginning. Examples are scattered throughout this text, and sources of them abound in the literature and support materials for teachers. One location of basic beginning information is NCTM publications, which have regular columns like “Technology Reviews,” “Publications,” and “Products.” Each of these columns, but particularly “Publications,” will give a basic description, along with an evaluation, of items submitted for review. Not everything that is appropriate as a resource is submitted, but the supply is adequate for a beginning. Another source of ideas is the “Problem of the Week Contest” found at locations like:

http://www.olemiss.edu/mathed/contest/contests.htm
www.umassd.edu/mathcontest/
http://www.whitehouse.gov/kids/math/index.html
http://mathres.kevius.com/problem.html
http://mathforum.org/

There are advantages to classifying games and tricks so they can be readily integrated into a curriculum. As you develop your collection, some will be your favorites. Those, you will probably be able to do reflexively and blend them into a class without much initial planning. “1089” is an example of a trick that is a good one to have as a reflexive presentation.

Write any three-digit number, without repeating the digits. That number is to be reversed (358 becomes 853) and the smaller subtracted from the larger. At this point, two directions can be taken, both of which amaze students and each of which holds additional teaching value.

First, tell the students that the tens digit (note subtle use of place value) of their missing addend (difference in subtraction) will be a nine. Careful observation may reveal individuals who may not have that in their answer. You have just gotten a diagnostic clue. Those individuals may not have heard or followed the instructions. It might also be that these individuals have difficulty dealing with subtraction involving regrouping. Any student experiencing difficulty here often can be prompted and assisted quickly to get over the hurdle, at least temporarily. The other direction can amaze students and lead them into discovering a pattern at the same time. Announce to them that if they tell you their ones digit, you will tell them the hundreds digit in their answer, or vice versa. They do not need to give the place, only the digit. Because the two digits have a sum of nine, determining the missing digit is quick and simple. The discovery part of this is valuable for the students. You do not want to tell the secret. Instead, lead them through the process of figuring it out. One crucial word of caution here: Students who determine the pattern need to be kept from telling.

There are two clues to help students discover the pattern. First, when a student says a number, “three,” for example, you say “un six.” Repeat that with a few more students, supplying different numbers. Those who “know” the trick can be encouraged to play along to help others or to think of clues that will not tell the others the secret. Generally, this “un” will help more students past the barrier and they will know what is being done. Some students may need more help. For them, and this is the second clue that is to be used if the first one fails, when a someone says “four” as the digit, you say “four un five.” Most students will reflexively say nine because they often say “four un five” to represent four plus five. Now almost everyone should know the secret. Normally, this discourse only takes a few minutes. Those who know the secret and play along will frequently give a reversal as a part of the examples; that is, if one student says two and you say “un seven,” the knowledgeable student will say “seven” allowing you to say “un two.” This too can be a very strong hint for those having difficulty.

If a student says the digit is nine, you say you know it is in the ones place. Then you need to instruct any students having a 9 in the ones place to write a zero in the hundreds place of the missing addend (giving a missing addend of 099), even though that is not standard procedure. Having a zero in the hundreds place is necessary for the next stage of the trick.

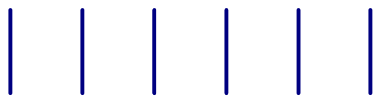
Continuing the trick, take the subtraction answer, whatever it is, and reverse it, adding the two values. Using 853 – 358, the missing addend (difference) is 495. Reversing gives 594. Adding 495 and 594 yields 1089. This will be the sum in all situations.

A common question is whether or not this will work all the time. In response, have the students do another example and form a conjecture. More than likely, they will not think to consider the fact that each student selected different values initially. Because everyone got the same answer it is probable that the procedure works for all examples. In the process of doing another example, the students are again practicing skills they need. If necessary, lead them to do still another example. Eventually they will conclude that the procedure works consistently, barring arithmetic errors.

The elaborate explanation was used with “1089” to give you insight into how to relate number tricks to a class. The rest of the number tricks discussed here are brief because it is assumed you will be able to insert the diversions, elaborations, and questions needed to attract student attentions. The examples given here are ones that are known to appeal to students. These are only a small part of attractions for students that are available.

Divide 30 by $\frac{1}{2}$ and add 10. What is the answer?

Given six congruent parallel line segments arranged so they are perpendicular to an imaginary horizontal line, add five more line segments to make nine.



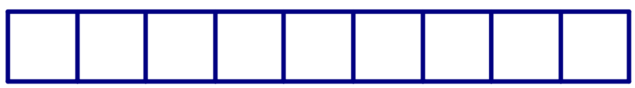
Where does the “Z” go and why?

A EF HI KLMN T VWXY

BCD G J OPQRS U

What value can be added to 1,000,000 so the result will be larger than if the 1,000,000 is multiplied by the same value? Is your answer unique? Why or why not?

Given a nine-stall-long horse barn, how can ten horses be housed such that no two horses occupy the same stall, none are running free, none are being ridden, and so on.



Pick a number. Triple it. Add 12. Divide by 3. Subtract 4. What do you get? Why does this work?

Select a three-digit number (374). Affix a duplicate of that number to either end, giving a six-digit number (374,374). Divide the six-digit number by 7. Divide the missing factor from that division by 11. Divide the missing factor from that second division by 13. What did you get? Why does this work?

Some movie scripts describe ransom situations where a million dollars in small unmarked bills is to be left at some remote location. The alleged culprit picks up the package and runs away. Suppose the payment was in $10 bills. How much would the $1,000,000 weigh? A bill in United States currency weighs approximately 1 gram. What is a reasonable denomination for the alleged culprit to request so the escape can be effected?

A window was a square, a meter on a side. That window admitted too much light so half the area was covered. After that, the window was still a meter high and a meter wide. How can that be?

Use three line segments (straight, curved, open, or closed) to divide a circle into eight sections, each of which has the same area.

A bear was followed 3 miles south, 3 miles west, and then 3 miles north. At that point, the trail crossed the initial trail. What color was the bear?

Suppose the earth is a sphere with an equator 25,000 miles in length. A circular band is placed around the equator that is concentric with it and 25,000 miles plus

10 feet long. Which of the following most closely approximates the distance between the band and the equator:

(A) Thickness of standard piece of notebook paper
(B) 8.5-inch side of standard piece of notebook paper
(C) 11-inch side of standard piece of notebook paper
(D) You

A “hole” can be cut in an 8.5-inch by 11-inch piece of paper that can be large enough for a person to step through. How?

This is a sample of the viable candidates for a collection of tricks. The collection of ideas can come from classes, reading, conferences, texts, conversations with peers, and submissions by students. Once you start giving these to students as challenges, they will provide you with additional examples. A select few elements from your collection will become your favorites, and you will be able to do them reflexively. Others will mandate that you review as you plan to use them. This last statement implies the need for some organizational scheme that will permit you to find items appropriate for a given class or topic. It is important to develop some method of organization at the start of building your collection. Otherwise, the collection becomes too unwieldy and, as a result, is not used because items cannot be located. Email Doug Brumbaugh at brumbad@pegasus.cc.ucf.edu if you want a collection of several hundred games, tricks, and activities.

PROBABILITY AND STATISTICS

The concepts relating to probability and statistics are usually presented through laboratory activities, which is great. Typical activities for these topics involve cards, dice, coins, spinners, population bar graphs, and circle graphs for budgets. Traditionally, probability and statistics are joined, and yet there is ample opportunity to do one without the other. In society, the level of inability to deal with or understand topics relating to probability and statistics is much higher than it should be. Part of the reason lies in lack of exposure. Ironically, there are varied and interesting activities that can be done in a classroom that will attract students to probability or statistics, especially with the availability of technology. For example, have each student bring in at least 20 pennies for a class. These pennies should be gathered as change between the time the assignment is made and the time when the pennies are to be brought. Explain to the class that pennies are minted each year and ask them what they think the distribution of minting years will be. The general guess is that the mintings will be distributed over the past several years, with the possibility that as the difference between the current year and the minting year increases, the number of pennies will decrease in that category. Then, in class, have each student sort the pennies by minting year and record the results. Compile the totals. Usually about two thirds of the class total will have been minted in the last 2 years. Why is that? Piggy banks! A significant problem for our government stems from the fact that people tend to save pennies, taking them out of circulation for long periods of time. Thus, the supply needs to be replenished regularly and most of the pennies in circulation are relatively new.

The statistical potential for this activity is impressive. Students can be asked to do some, or all, of:

- Gathering data
- Formulating hypotheses
- Organizing material
- Classifying results
- Compiling facts
- Analyzing information
- Presenting findings
- Investigating reasons for the happenings

There are other extensions of this activity into the classroom environment, but the preceding list is a beginning. Consider the impact of an activity like this on the classroom setting. The request is not typical, which will generate discussion and some interest. The results are not what would normally be expected, which heightens awareness of the impact of personal activities. Most significantly is the affective influence. Students will generally laugh and be somewhat surprised by the results. Save the data. At some later date, when doing spreadsheets, for example, reproduce the data along with saying, “Remember when we gathered the pennies . . . ?” That question will generate the emotional highs that were produced in the class at the time the activity was done. You capitalize on prior positive experiences and use them to enter a new lesson with a positive attitude. (Credit for this idea goes to Jim Rubillo, NCTM Regional Conference, Boise, Idaho, October 8, 1994.)

Surveys are a typical statistical happening as a part of the study of mathematics. Questions like the number of siblings, pets, CDs, TVS, pairs of tennis shoes, and so on are common. Traditionally the information is compiled using frequency distributions and graphs are produced. That in itself is a good activity if for no reason other than it makes a connection with the real world. Many publications contain graphs to represent data, probably the most notable of which is USA Today. Incidentally, this paper provides a multitude of teacher help items, free, and they are on the Internet. Regrettably, the emphasis almost always focuses on the graph production. The lack of connection comes from the idea that most often, in the real world, graphs are not produced but, rather, interpreted by the consumer of the information. Rather than emphasizing interpretation of graphs from data, the accent should be on interpretation.

A statistical topic often overlooked in surveying is sampling. There is a need to consider how well the sample represents the population. The media often report results a survey as “67% of the population with a 5% sampling error.” If you do not believe sampling techniques adequately represent a situation, then the next time you have a blood test, you will need to give all your blood, not a sample.

Some sampling techniques are far from being random and yet the results are often presented in a manner that implies they represent the population. For example, suppose you conduct a survey where you interview a number of teenagers exiting an upscale clothing store. You ask them the amount of their monthly allowance. Do you suppose that is going to be representative of the allowance of all teenagers? Ironically a reporter for a paper recently did interview students coming out of an upscale clothing store and then produced a featured article about student allowances being over $100 a week. That means some of these teenagers were getting an allowance that is about 20% of a starting teacher's salary.

Discussions and activities dealing with statistical treatment of the information generated are essential in today's data-rich society. As members of the public, we are bombarded with statistical information, claims, "research"-based statements, and so on. Many people assume that if an article, advertisement, or statement contains numbers or comments about the statistical impact of the values given, that the remarks are factual. That is not a safe assumption to make. Often incorrect statistical procedures are used to generate the claims being made. Certainly there are cases where the "incorrect use" is intentional to "fool" the public, but the sad fact is that many times, the individuals using the statistical treatment might not realize an error is being made. Thus, incorrect images are portrayed to consumers of the data and information.

That last paragraph is a beginning, perhaps a plea, for you to campaign for additional statistical topics to be inserted into the mathematics curriculum. We live in a data-rich society and as consumers, must be aware of what is being said and how it is construed to avoid false impressions. IF (note the big IF) you do start campaigning for statistical insertions into the mathematics curriculum, you are placing the onus on you at the same time. More than likely your statistical background is limited. That certainly is not sufficient background on which to base teaching extensive statistical topics.

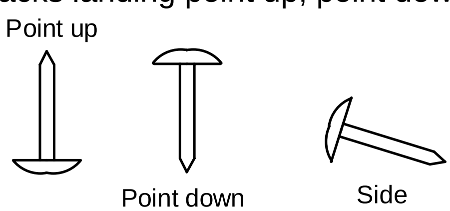
Statistics topics can be inserted into the curriculum anywhere it is appropriate to include a real-world example. The problem is that such an insertion would require a greater knowledge of statistics than most people have. We are talking concepts that would go beyond mean, median, and mode. Probably the most important statistical notion is the understanding of sampling procedures. We do not mean the technical treatment of the topic but simply the basics of random sample being the underlying requirement of all statistical procedures. That could easily be taught when the idea of bar graphs and histograms are covered.

For example, after a shooting occurs in a school that is not necessarily local, reporters will survey local parents who are picking their children up after school. Generally the question deals with whether or not the parents feel their children are safe. With the gathered data, the local media present their report as if they have a feeling of the local parents. The interviewees could have been randomly selected from the parents who were picking up their children. However, the parents surveyed were not a random sample of the parents whose children attend the school. So, we get statistics and a report that sounds true and representative. Very few of the recipients of the report will have even the slightest indication that the information is far from representative of the true situation. Most significantly, a little statistical education could fix most of that.

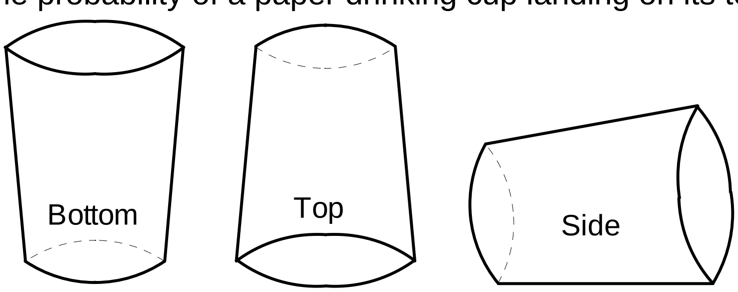
If the statistics teaching is essentially a list of topics inserted into the curriculum, where will they be placed?

Do students have adequate background to study statistics? There is a need for algebraic understanding to adequately deal with even basic statistical topics. That creates a dilemma though. Many students opt out of mathematics as soon as they can. Suppose for the sake of this discussion that they all have a minimal familiarity with the topics generally found in an Algebra I course in the secondary program. Is that an adequate background for them to take a serious statistics course? If we do not offer (require) a statistics course for the masses, are we creating a statistical elite? Is it wise to have a society with statistically elite individuals coupled with the multitudes who are essentially clueless? What is the role of statistical programs like Fathom (www.keypress.com)? What is the role of the graphing calculator?

Probability activities generally focus on coins, dice, and maybe cards. Students are generally familiar with the outcomes here. Determine the probability of thumbtacks landing point up, point down, or on the side.



What the probability of a paper drinking cup landing on its top, bottom, or side?



Students initially indicate that the three positions will be approximately equal in frequency. A little discussion, often with no prompting from the teacher, leads to the conclusion that the side might be a more common landing position for the cup than either the top or bottom. Still, the question is not answered and it is time to DO the activity.

After doing the cup activity, it is possible to bring bias into the discussion. From a statistical standpoint, the term means getting results that are not typical because of something that has been done to influence the outcome. Alter the cup in some manner so it will always land on the bottom. Because students typically think of putting a weight in the bottom, eliminate that as a possible method of biasing the results. There are several creative results someone will probably employ:

- Remove the bottom
- Cut close to bottom so only bottom and small part of side is left
- Make “parallel” cuts from top to bottom around cup
    - Fold each strips at bottom end so strip is parallel to bottom
    - Result looks like helicopter blade

A typical statistics approach is to investigate an athlete’s salary and claim to fame. Suppose a baseball player is paid $14,000,000 a year and is famous for hitting home runs. The player does more than bat and has hits other than home runs but, if home run hitting is the skill that merited the pay, and if the player is on the team to do that, how much is each home run worth? This idea could be extended to include all hits and pay per hit, or pay per base, catch, throw, and so forth. Continuing with the hitting example, a person with career batting average of 0.300 is almost guaranteed a place in the Baseball Hall of Fame. Do you suppose the batting success rate is given as 0.300 rather than 30% for affective impact? But a 0.300 average means a 70% failure rate. What would be thought of a surgeon who was “batting 0.300"?

Begin class by presenting a large box or bag that hides the contents from the students’ view. Discuss that you will be covering the gathering of statistical information today. Tell them they will be classifying the contents of their data package into six categories: T, B, G, Y, R, and 0. Mention that they will be permitted to keep the data package at the conclusion of the activity, doing with it as they wish. The object is to make the discussion sound as much like so many typically boring starts to a mathematics lesson. Only when you are well into lulling them to sleep do you produce the packages of M&Ms and conduct the activity headed toward an objective of your choosing.

NUMBER THEORY

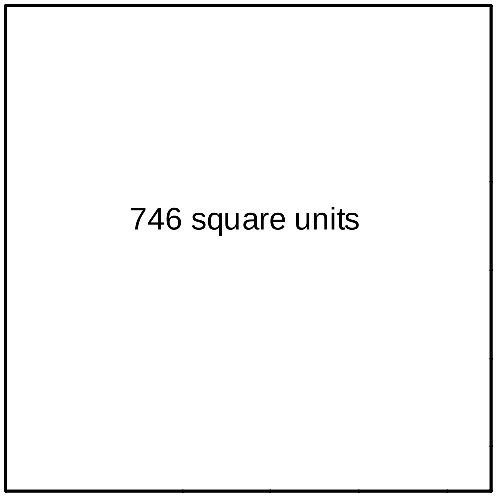
Number theory is one of those names that can mean different things to different people. Ideas dealing with prime, composite, perfect, abundant and deficient numbers, divisibility rules, greatest common factor (GCF), least common multiple (LCM), factors, multiples, basic proofs, and number oddities would all be included.

The terms factor and multiple are essential ingredients of number theory. Amazingly, they are difficult for many students to understand, even those who typically do well in mathematics. Many students are able to parrot the definitions, but there is limited comprehension of what is being said. When a task is to find the LCM of 10 and 12, students become confused because they do not know whether the multiples of 12 are 12, 24, 36, . . . or 1, 2, 3, 4, 6, 12. Furthermore, total agreement does not exist within the mathematical community on multiples. Some say 0 is a multiple of 12 and others say it is not. Consider the dilemma this causes for students, especially if one text they use says 0 is a multiple of 12 and the next says it is not. A result of this confusion and inconsistency is students who are not certain of how to apply definitions. In that mode, their ability to deal with the task of finding the LCM is hampered.

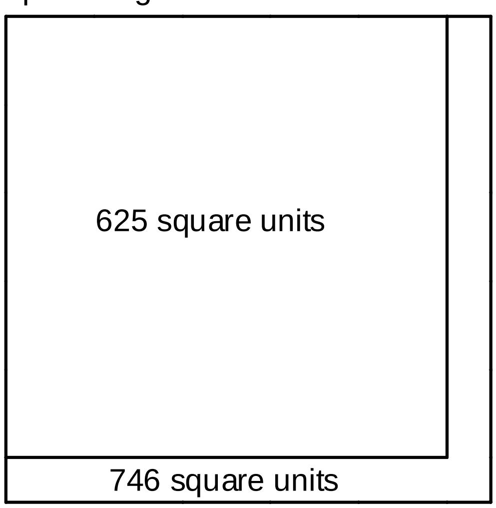
Still, number theory holds a multitude of opportunities in the world of mathematics at all levels. Early number theory was dominated by arithmetic that emphasized geometric measurement and calculation. In the school curriculum, counting numbers are classified as odd or even and also as one, primes, and composites. Primes and composites are encountered in their own right and extended as useful tools in simplification of fractions, LCM, GCF, multiples, and factorization of numbers. Perfect, abundant, and deficient numbers are also based on factoring. Six is a perfect number. The factors of 6 are 1, 2, 3, and 6. Excluding 6, the sum of the remaining factors is 6. Twenty-eight is the next perfect number. Excluding 28, the sum of the other factors (1, 2, 4, 7, 14) is 28. From these two examples (even though it is a dangerous practice to base arguments on such a limited exposure) you should conclude that a perfect number equals the sum of its factors, excluding the number itself. In the same manner, 8, 11, 15, and 25 are deficient numbers whereas 12, 18, 24, and 36 are abundant numbers.

Thanks to technology, finding the square root of a number is not nearly the challenge it was in the past. Today, a student need only enter 746 into most calculators and press the "$\sqrt{\ }$" key to view the desired answer. Doing such a computation by hand is not an easy task, especially if you want it to the nearest thousandth. Some rationalize the need to continue doing such computations by hand so there is an understanding of the true meaning of the result. Do you think doing things manually provides insight we would call understanding? The process of finding a square root can be explained geometrically, giving the students a much greater appreciation for the operation. When computing the square root, it must be realized that the answer is the length of the side of a square having an area equal to the number used to start the computation, 746 in this case.

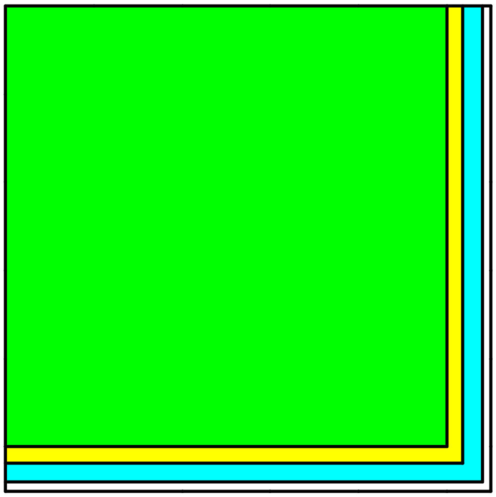
This square has an area of 746 square units.



The task is to determine the length of the side of that square. Assuming the same unit, a square of area 625 is known to be smaller than a square of area 746. Inserting the smaller square inside the larger, shows the side length of the larger square is greater than 25 units.



Even larger squares can be placed inside the initial square of area 746. In each case, the length of the side of the square being inserted is known and is smaller than the length of the side of the initial square. This process can be repeated as many times as necessary, getting closer and closer to the desired value.



The three figures give a geometric interpretation of finding the square root of a number. Thinking algebraically and looking at the last figure, it is known that the total area is 746 square units and a square with side length 25 square units has been inserted in the top right corner. What is the area of the region between the inserted square and the initial square?

$746 = (25 + X)(25 + X)$

$746 = 625 + 50X + X^2$

$746 - 625 = 50X + X^2$

$121 = 50X + X^2$

$121 = X(X + 50)$

At this point the student would be asked to approximate a value for X. Suppose the estimate is 2. An estimate of 2 units for the value of X would mean that a new square is created, and its side length would be 27. This would be shown in the innermost “backwards L.” Algebraically, the new situation would be

$746 = (27 + Y)(27 + Y)$

$746 = 729 + 54Y + Y^2$

$746 - 729 = 54Y + Y^2$

$17 = 54Y + Y^2$

$17 = Y(Y + 54)$

The student would now be asked to approximate a value for Y.

The next “backwards L” shows 0.3 being used as an approximation for Y. The process can be continued, approaching the value for the square root of 746 as closely as desired. Blending the pictures with the algebraic explanation clarifies the concept for most students.

INTEGERS

Students first formally encounter integers late in the elementary grades or in the beginning of their middle school years. In many instances, they have been proceeding through their mathematical careers totally unaware that there were any numbers other than wholes, fractions, and decimals. Integers seem so foreign at first. We discuss positive integers and elaborate on the need to express 2 + 3 = 5 as $^-2$ + $^+3$ = $^+5$. In most instances it is difficult to rationalize this to the students, but usually, after enough insistence from us, they succumb. The negative integers along with the other operations are then introduced over a period of time. Somewhere along the developmental line, we abandon the demand that $^+2$ + $^+3$ be written as $^+5$ and permit them to revert back to writing it as 2 + 3 = 5. We continue the rationalization by saying, “Everyone knows that if no sign is present, the direction is positive.” Some students have to wonder why we made such a big deal of the situation if we are so willing to abandon it. Is it any wonder students question why we ask them to learn things at times?

With calculators, students may discover integers. Experimentation might lead to discovering that 7 - 3 produces an answer different from 3 - 7. Certainly this experimentation depends on observation skills of students because they need to notice the presence of the “-” associated with 3 - 7. The calculator and patterning skills can be used to assist students in “discovering” the various rules of operation for integers. How many examples of the type $^+5 + (^-3) = {}^+2$ would need to be done on a calculator before a student determines that the second addend is actually subtracted from the first when the first addend is positive and the second is negative? Assistance is provided to the discovery process through a carefully developed set of problems that guide the student to appropriate responses. In this case, the problems would all have the first addend positive, the second negative, and the absolute value of the first greater than that of the second addend.

Patterning can lead a student to conclude that the product of two negative factors is positive. Appropriate readiness skills must be assumed. A sequence of problems would start:

$(^+5)(^-4) = {}^-20$
$(^+4)(^-4) = {}^-16$

After this second step, call attention to the following ideas:

The first factor decreases by 1 when the second problem is compared with the first.

The second factor stays the same in both problems.

The product increases by 4 when comparing the two problems.

The sequence of problems will be continued, each time comparing the new problem with the last and amplifying the observation that the first factor is decreasing by 1, the second stays the same, while the product increases by 4.

$(^+3)(^-4) = {}^-12$
$(^+2)(^-4) = {}^-8$
$(^+1)(^-4) = {}^-4$
$(0)(^-4) = {}^-0$
$(-1)(^-4) = {}^+4$

Plop! How can this be? If the pattern being followed to this point is true, it must be the case that a negative times a negative is a positive. The key to having the appropriate impact in this lesson is to make the sequence of problems long enough so the students reflexively say with appropriate boredom for each new problem shown that:

> "The first factor decreases by one."
> "The second factor stays the same."
> "The product increases by four."

If necessary start with a larger first factor to achieve the desired impact.

Another process can be used to provide indicators for students as to the outcomes of finding products of signed numbers. A video camera and associated VCR and large monitor are necessary for this activity. The students will be videotaped walking. In all environments of this activity, forward motion is defined as being positive and backward motion is defined as being negative. Students first are filmed walking forward, indicating they are walking forward by extending arms in front of them, wearing ball caps with peaks pointing to the front, and so on. They can walk in rows, columns, groups, and so on. Positive motion could be climbing up a tree (with appropriate consideration for risk). Playing the tape at this point by running it forward would contribute a second positive factor. The result on the screen would show students walking forward, a positive product produced by two positive factors.

Next, play the tape in reverse. The students are indicating forward motion, but the result on the screen shows them walking backward. Thus, a positive factor (forward walk) and a negative factor (reverse tape play) yield a negative product. Negative motion can be indicated by walking backwards, going up a sliding board, and so forth. Indication of the direction while being filmed is important, such as with hats pointing back. When the tape is viewed, walking backwards with the tape running forward yields an image on the screen going backwards. Thus, a negative factor coupled with a positive factor yields a negative product. All of the preceding activities are valuable in their own right, but they are background work for helping students conclude that two negative factors yield a positive product. This can be demonstrated by playing the tape backward on a section where the students were walking backward when the tape was made. The result on the screen will be forward motion. Thus, two negative factors yield a positive product!

The concept of videotaping can be extended to division as well. Students walk, but no indication of direction is given. When the tape is played, if there is forward motion on the screen and if it is known the projection unit is running forward, a positive product and one positive factor are present. The students then deduce the missing factor must be positive because the only way a positive product is generated is from either two positive factors or two negative factors. The rest of the division situations can be reasoned in a similar manner. This can be extended to a generalization that most students are not aware of. Dividing with like signs yields a positive result. Dividing with unlike signs yields negative results. Students are amazed to notice these generalizations are the same as those for multiplication.

Mnemonics are wonderful tools for helping students remember various material. When dealing with multiplication of signed numbers, one helpful statement is, "It takes two minus signs to draw a plus sign." Another helpful hint that students enjoy for remembering the rule for multiplying two signed numbers comes from the following story involving a "thing" (or happening), "person," and "moral." In this story, good is represented by "+" and bad by "-."

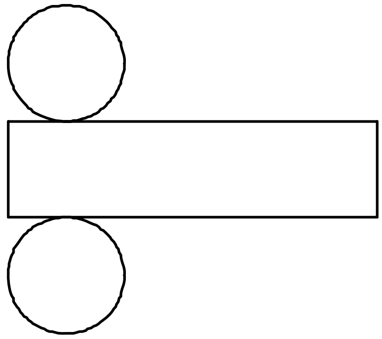
Thing	Person	Moral	
+	+	+	(Good thing to good person is good)
+	–	–	(Good thing to bad person is not good)
–	+	–	(Bad thing to good person is not good)
–	–	+	(Bad thing to bad person is good)

Stories, like the one about moral, have to fit within the tolerances of your personal construct as well as that of the school and community, but they are cute, and they do help many students remember rules. Mnemonics are not overly necessary today in situations like finding the product of signed numbers. Use a calculator. They give the proper sign every time, assuming the information is entered correctly.

GEOMETRY READINESS

It is important to insert as much geometry as reasonably possible into the general mathematics curriculum. For one thing, the subject offers practical applications in the real world of students: skateboards and curb jumping, in-line skates, balls for different sports, out-of-balance tires, and so forth. Technology that works with geometry (most notably Cabri, Geometer's Sketchpad, the Casio A-21S/A-22T and the TI-92) brings in almost limitless opportunity for exploration. Students typically like the concept of geometry. We work against that in many instances by trying to formalize things. Let them investigate their curiosities. Show them where the things they intuitively know can be applied. Do not force them to proofs too fast. Let the natural and historical order of events take over wherein they see a situation, question it, convince themselves with sketches and pictures that it is or is not true, and then, after all that, begin to prove it.

There are a multitude of confusion points and contradictions in a student's world of geometry. We define line segment as having a definite beginning and end and then tell students to write the definition on "lined paper." A point has no length, width, or thickness and then we call little blobs, on the paper, board, overhead, or screen, points. They hear "pyramid" and think Egypt. Then we tell them about right and non-right pyramids as well as pyramids that do not have square bases. Area is a two-dimensional idea, but then they find the surface area of a solid. Oh yes, we know that each of those faces is on a plane and, thus, it really is a two-dimensional figure, for that part of the solid, but they do not always make the connection. Then, when we want to find the area of a right circular cylinder, the explanation involves slitting the cylinder with a cut perpendicular to the base and “rolling” it out so the cylinder becomes a rectangle with two circles attached.



Examples similar to those in this paragraph abound. Sometimes they confuse students and sometimes they are oblivious to them. The problem is, we are never certain which way they will be impacted.

ALGEBRA READINESS

Standard 5 of the NCTM Standards bears the title Algebra, as does Standard 2 in Standards 2000. Both publications portray algebra as a topic that is to be a part of the core curriculum taken by all students. The Standards introduction is very careful to explain that this is not necessarily the algebra course as described in prior years of school mathematics. Standards 2000 integrates algebra into the curriculum throughout the K–12 experience. The expectation is that all students will be competent with concepts like variable quantities, expressions, equations, inequalities, and matrices; use of tables and graphs; operations on expressions; matrices as a solution tool for linear systems; and performance of transformations based on the theory of equations. This list represents a continuing study of algebra throughout the high school years, but demands for preparation are placed on the prior years as well.

Background for algebra begins with the study of arithmetic in the primary grades. Even with a problem like 7 - 4 = ?, the ? becomes a representative of a variable. The Cuisenaire rods, with their letters representing the name of the rod, show another place where students will have formative background for the idea of variable. Still later, they encounter formulas for area and perimeter that have letters representing numbers. Again, variable is there, but not stressed. Why emphasize operations on fractions so heavily in the curriculum? One reason is for fractions in their own right, but another focuses on background skills for algebra situations.

Algebra is the language through which most of mathematics is communicated. It also provides a means of operating with concepts at an abstract level and then applying them, a process that often fosters generalizations and insights beyond the original context. (NCTM, 1989, p. 150) Students need to explore algebraic concepts in an informal way to build a foundation for the subsequent formal study of algebra. . . . Students should be taught to generalize number patterns, to model, represent, or describe observed physical patterns, regularities and problems. These informal explorations of algebraic concepts should help students to gain confidence in their ability to abstract relationships from contextual information and use a variety of representations to describe these relationships. (NCTM, 1989, p. 102)

The preceding description taken from the Standards speaks to information necessary for the study of algebra. Call it general mathematics, pre-algebra, informal algebra, or fundamental algebra, it is background work. These are the essential ingredients all students must possess prior to entering a formal study of algebra. A similar discussion could be developed for geometry, probability and statistics, number theory, problem solving, thinking skills, measurement, mathematics as communication, estimation, patterns, and number relationships. The bottom line is that all this background material must be developed for students prior to the more formal high school core curriculum.

MATHEMATICAL ILLITERACY

How is mathematics education viewed by society? Tell someone you are studying to be an engineer and they will say, “Wow.” Tell that same person you are going to be a teacher of mathematics and you may hear something like, “Why?” The youth of our country are our future! Why not encourage our best and brightest students to enter the teaching profession? Society does not always seem to think very highly of the study of mathematics. “Innumeracy by John Paulis [1987, New York: Hill and Wang] describes the crisis in mathematics education as similar to the crisis in literacy for the American public - - too many people know too little.”

Are people aware of the consequences of mathematical illiteracy? No. Headlines often play on the importance of keeping up with the rest of the world scientifically, which is important, but which most people see as someone else’s responsibility. Unfortunately, the consequences of mathematical illiteracy reach well beyond our conquest of space, the development of new chips for computers, etc. (Cozzens, 1989, p. 2)

After reading that you should say something like, “Somebody should do something about that.” You’re it!

CONCLUSION

We have discussed many facets of mathematics in general (or whatever you want to call it). There are many issues that have not been discussed. You should be thinking about the list of items that have not been discussed that need to be included in your courses.

Resources

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