

Problem Solving Exercise Solutions

Describe at least one situation that does not have one right answer. Your discussion should include all the potentially correct responses. For example, what route do you travel to get to school? Many paths will lead to the same conclusion!

Solution: answers will vary.

You are giving a party. You are not a guest.

First doorbell ring brings one guest.

Second doorbell ring brings 3 new guests.

Third doorbell ring brings 5 new guests.

Etc. Each new doorbell ring brings 2 more new guests than last one.

How many guests enter on the 20th ring?

39. The numbering is always one less than two times the doorbell ring number. The first one is $(2)(1) - 1 = 1$. The second is $(2)(2) - 1 = 3$. The third is $(2)(3) - 1 = 5$. And so on.

What is the grand total number of guests after the 20th ring?

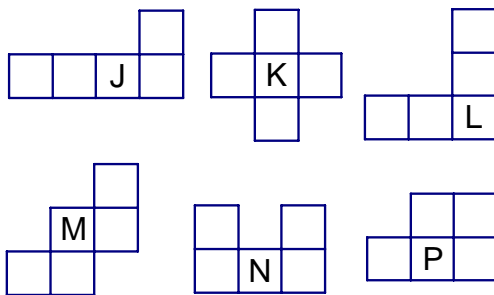
400. After the first ring there is a total of 1 person. After the second ring, the total is 4 (1+3) people. After the third the total is 9 people (1+3+5). There is a pattern, 1, 4, 9, . . . , perfect squares.

Are generalizations possible?

Sum of first n consecutive odd counting numbers = n^2 .

Sum of first n consecutive counting numbers = $\frac{n(n+1)}{2}$

List at least three additional questions a student could generate using these shapes.



Solution:

Which shapes make a cube? None since there are only five faces and cubes need six congruent faces.

Are there other ways these squares can be put together? Yes.

You do have to decide if mirror images, rotations, and so on are acceptable as being different.

Can these pieces be put together to make a big square? No.

There is a total of 30 squares with the six pieces, and 30 is not a perfect square.

Can five of these pieces be put together to make a square? No.

While five pieces would give 25 unit squares, and 25 is a perfect square, any five given pieces cannot be used to make a square.

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What is the maximum perimeter that you can get using five congruent squares? 20 units because all edges are exposed.

What is the minimum perimeter that you can get using five congruent squares? 10 units, as shown in figure P. This is because the least number of edges are exposed.

Use another geometric shape to develop questions along the line of thinking developed with the example of five congruent squares.

Solution: Answers will vary. You could have flat patterns of:
two congruent circles and a rectangle configured to make a cylinder
four congruent triangles to make a tetrahedron
Etc.

Describe how nine blocks, where eight are known to weigh the same and one is heavy, require two weighings to identify the heavyweight. Can this be done more than one way?

Solution: Put three blocks on each side. If they balance, the heavyweight is one of the remaining three blocks that are not in use. The total number of weighings is two.

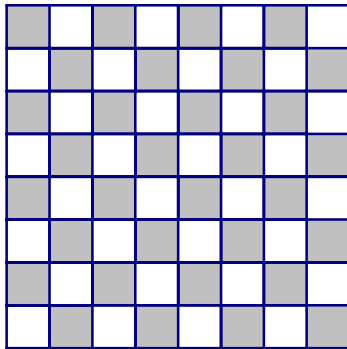
With three on each side of the balance shows unequal loads, then it is known that the three blocks not in use are all light and do not need to be considered. Furthermore, the heavyweight is one of three blocks and it can be identified in a second weighing. The total number of weighings is two.

NO, there is no other way to solve this situation.

The Tower monk story makes a wonderful mathematics problem. Use a set of six disks and do the puzzle in 63 moves. Then try to do the puzzle in 63 or fewer seconds. How did you do?

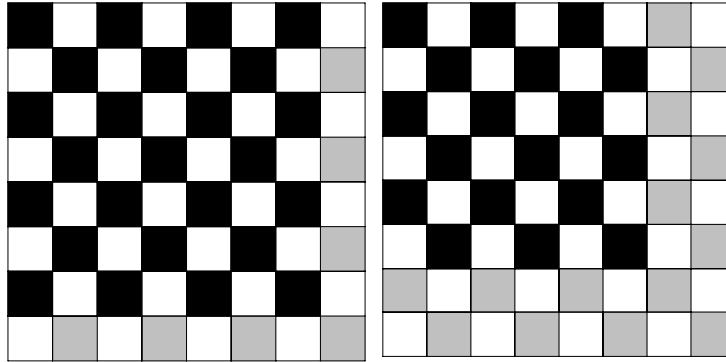
Solution: 63 is the minimal number of moves. Good luck on 63 seconds!

How many squares are on a checkerboard?



Solution:

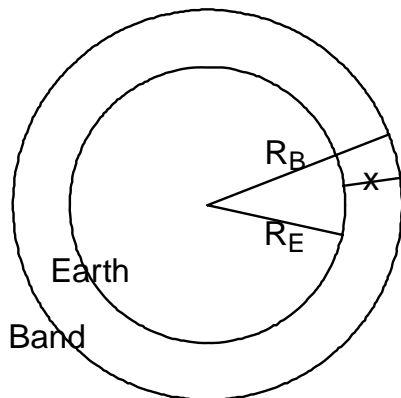
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8x8 --> 1, 7x7 --> 4, 6x6 --> 9, . . . 1x1 -->64 (204)

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Suppose the earth is a sphere and the circumference at the equator is 25,000 miles. A band is placed around the earth, concentric with the equator. The circumference of the band is 25,000 miles + 10 feet. A standard sheet of 8.5 - by 11-inch notebook paper is about 0.003 inches thick. Would that paper fit between the band and the equator in any of its three configurations (thickness, width, length)? What is the thickest thing that could fit between the band and the equator?



- R_E = Radius of the Earth
- R_B = Radius of the Band
- $x = R_B - R_E$
- C_E = Circumference of the Earth
- C_B = Circumference of the Band
- $C_E = 25,000$ miles
- $C_B = 2\pi R_B$
- $C_B = 2\pi(R_E + x)$
- $C_B = 2\pi R_E + 2\pi x$
- $C_B = C_E + 10$ feet
- $C_B = 2\pi R_E + 10$ feet
- So, $2\pi R_E + 2\pi x = 2\pi R_E + 10$ feet

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Kid 4	O	C	C	O	O	C	O	O ...
Kid 5	O	C	C	O	C	C	O	O ...
Kid 6	O	C	C	O	C	C	O	O ...
Kid 7	O	C	C	O	C	C	C	O ...
Kid 8	O	C	C	O	C	C	C	C ...

OR, look at factors of each number

Primes and composites (non-square) have even number of factors

Implication -- each ends up as things started

Squares have odd number of factors

Implication -- each of these opposite of as things started

Folklore describes a young man and a king's daughter. They were very much in love and wanted to marry. The king did not want the wedding to occur because the young man was not of the appropriate breeding, and besides, the young man did not possess the wealth to provide the lifestyle to which the girl had become accustomed. The young man was a clever fellow and offered the king an opportunity that was too good to pass up. The boy offered to go away if the king would be willing to give him some grain. The king would place one kernel of grain on a square of a checkerboard. The second square would contain two kernels of grain, the third four, and so on, until all 64 unit squares on the checkerboard were used. The total of the kernels of grain would be the young man's wealth and with it, he would leave, unless the king deemed otherwise. The king accepted the challenge. Do you think the young man married the king's daughter?

$2^{64} = 1.84 \times 10^{19}$ or 18,400,000,000,000,000,000 kernels of grain, just on the last square. The one before it would contain 9,220,000,000,000,000,000 kernels of grain. The young man ended up possessing more grain than the world could produce in several years.

A sheet of paper is folded in half, and then in half again, and again, and so on, until a total of 50 folds is made. The question is, "How high is the stack of paper?" The assumption here is that the 50 folds can be made.

Typical paper is 0.003 inches thick.

Stack of paper is 2^{50} thick x 0.003 $\approx 3.3776 \times 10^{12}$ inches thick.

$$\text{Stack} \approx \frac{3.3776 \times 10^{12} \text{ inches}}{\left(\frac{12 \text{ inches}}{\text{foot}}\right) \left(\frac{5280 \text{ feet}}{\text{mile}}\right)} \approx 53,309,655 \text{ miles high!}$$

Use paper that is 0.004 inches thick and the stack is 70,901,841 miles high.

What is the largest number you can write using only three digits? The digits may be repeated. The first answer is typically 999, but after some thought you hear 99^9 or 9^{99} . This leads to a discussion as to which is larger. This is a wonderful opportunity to use rounding skills. We know 9 is close to 10 and 99 is close to 100, so we will round in both cases. Thus, 99^9 becomes 100^{10} and 9^{99} is close to 10^{100} . We know 100^{10} equals $(100)(100)(100) \dots$ ten times or 1 followed by 20 zeros. Similarly, 10^{100} is $(10)(10)(10) \dots$ one hundred times or 1 followed by

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100 zeros. Thus, 9^{99} is larger than 99^9 , by quite a bit. But, the real answer is 9^{99} and the result is huge! The problem is worked from the top down, so the base 9 would be taken to the $387,420,000^{\text{th}}$ power. The resultant value is about 500 miles long if typed using a 10 pitch font.

A customer enters a 7-11 convenience store and selects four items. The clerk informs the customer that the total cost of the four items is \$7.11 (excluding tax). The customer was amazed that the cost of the items was the same as the store name. The clerk informed the customer that the price of each item was multiplied to arrive at the total. The customer calmly informed the clerk that the prices should be added, not multiplied. The clerk obliged and the sum was \$7.11 for the same four items. What was the exact cost of each item (no rounding is necessary to obtain the exact values)? NOTE - - this proof is beyond many college students, but the problem is common. I list them both here mainly for your reference.

Solution: 7-11 - - This proof was developed by Dr. Howard Sherwood who is retired from the mathematics department at the University of Central Florida.

Find $x, y, z,$ and w such that $(x)(y)(z)(w) = 7.11$ and $x+y+z+w = 7.11$, where $x, y, z,$ and w are exact prices of objects.

Let $X=100x, Y=100y, Z=100Z,$ and $W=100w$.

Then $X, Y, Z,$ and W are integers and

$$(X)(Y)(Z)(W) = (3^2)(79)(2^6)(5^6) \quad [1]$$

$$X+Y+Z+W = 711 \quad [2]$$

Since 79 is prime and 79 divides $(X)(Y)(Z)(W)$, we must have 79 divides one of $X, Y, Z,$ or W .

Without loss of generalization, suppose 79 divides X . Then $X = 79(X_1)$ and [1] and [2] become

$$(X_1)(Y)(Z)(W) = (3^2)(79)(2^6)(5^6) \quad [3]$$

$$(X_1)+Y+Z+W = 711 \quad [4]$$

$1 \leq (X_1) < 9$ and (X_1) divides $(3^2)(2^6)(5^6)$ so $(X_1) = 1, 2, 3, 4, 5, 6, 7,$ or 8 . (we will eliminate all but 4)

Suppose $(X_1) = 8$. Then [3] and [4] become

$$(Y)(Z)(W) = (3^2)(2^3)(5^6) \quad [5] \text{ and}$$

$$Y+Z+W = 79 \quad [6]$$

At least one of $Y, Z,$ and W cannot be divisible by 5 otherwise 79 would be also. Thus 5^3 divides one of $Y, Z,$ or W and [6] is violated.

Thus, $(X_1) \neq 8$.

Suppose $(X_1) = 6$. Then [3] and [4] become

$$(Y)(Z)(W) = (3)(2^5)(5^6) \quad [7] \text{ and}$$

$$Y+Z+W = (7)(79) \quad [8]$$

Since 5 cannot divide one of $Y, Z,$ or $W, 5^3$ must divide exactly two of them (because 5^4 would be too big a divisor for any one of them).

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But then $Y+Z+W > (3)(79)$. Hence $(X_1) \neq 6$.

Suppose $(X_1) = 2$. Then [3] and [4] become

$$\begin{aligned} (Y)(Z)(W) &= (3^2)(2^5)(5^6) && [9] \text{ and} \\ Y+Z+W &= (7)(79) && [10] \end{aligned}$$

Since 5 cannot be a divisor of every one of Y , X , and W and since $5^4 = 625 > 553 = (7)(79)$, we can see that 5^3 must be a divisor of exactly 2 of the factors. Without loss of generalization, let $Y = (5^3)(Y_1)$ and $Z = (5^3)(Z_1)$.

Then [9] and [10] become

$$\begin{aligned} (Y_1)(Z_1)(W) &= (3^2)(2^5) && [11] \text{ and} \\ 125(Y_1)+125(Z_1)+W &= (7)(79) && [12]. \end{aligned}$$

Now, $(Y_1) + (Z_1) \leq 4$ otherwise [12] is false. If $(Y_1) + (Z_1) = 4$ then [12] becomes $W=53$ which contradicts [11]. So, $(Y_1) + (Z_1) \neq 4$. If $(Y_1) + (Z_1) = 3$ then [12] becomes $W=178$, which contradicts [11]. So, $(Y_1) + (Z_1) \neq 3$. If $(Y_1) + (Z_1) = 2$, then [12] becomes $W=303$ which also contradicts [11]. Thus $(Y_1) + (Z_1) \neq 2$.

Suppose $(X_1) = 5$. Then [3] and [4] become

$$\begin{aligned} (Y)(Z)(W) &= (3^2)(2^6)(5^5) && [13] \text{ and} \\ Y+Z+W &= (4)(79) && [14]. \end{aligned}$$

Since not all of Y , Z , and W can be divisible by 5 and none of them can be divisible by 5^4 , exactly one of them is divisible by 5^3 and exactly one of the others is divisible by 5^2 . Without loss of generality, let $Y=125(Y_1)$ and $Z=25(Z_1)$. Then [13] and [14] become

$$\begin{aligned} (Y_1)(Z_1)(W) &= (3^2)(2^6) && [15] \text{ and} \\ 125(Y_1) + 25(Z_1) + W &= (4)(79) && [16]. \end{aligned}$$

Since $(4)(79)=316$ and $(125)(3)=375$, it is clear that $(Y_1)=1$ or $(Y_1)=2$. If $(Y_1)=1$, then $25(Z_1)+W=191$ whence exactly one of (Z_1) and W is odd which in light of [15] would say that either 64 divides (Z_1) or 64 divides W . Clearly 64 does not divide (Z_1) because then $25(Z_1)+W > 191$. Thus, 64 divides W but then $(Z_1) = 1, 3, \text{ or } 9$ which leads to a contradiction. Thus, $(Y_1) \neq 1$.

So, $(Y_1) = 2$. But then [15] and [16] become

$$\begin{aligned} (Z_1)(W) &= (3^2)(2^5) && [17] \text{ and} \\ 25(Z_1) + W &= (2)(3)(11) && [18] \end{aligned}$$

From [17] and [18] it is clear that (Z_1) and W are both even.

But $(Z_1) < 4$ so $(Z_1) = 2$ whence $W=(3^2)(5^4)$ and $W=(2)(3)(11) - (25)(2)=16$, a contradiction. Thus, $(Y_1) \neq 2$ either, and in turn, $(X_1) \neq 5$.

Suppose $(X_1) = 1$. Then from [3] and [4] we get

$$\begin{aligned} (Y)(Z)(W) &= (3^2)(2^6)(5^6) && [19] \text{ and} \\ Y+Z+W &= (2^3)(79) && [20] \end{aligned}$$

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Clearly 5^3 must divide one of Y , Z , or W , but 5^4 cannot divide any of Y , Z , and W because if it did, the left side of [20] would be too big to equal $(2^3)(79)$. Without loss of generality we may write Y as $125(Y_1)$. Now 5 cannot divide each of Y , Z , and W so 5^3 must divide one of Z or W . Without loss of generality we write $Z=125(Z_1)$. Then [19] and [20] become

$$(Y_1)(Z_1)(W) = (3^2)(2^6) \quad [21] \text{ and}$$

$$125(Y_1)+125(Z_1)+W=(2^2)(79). \quad [22]$$

Since $\frac{(2^3)(79)}{125} = 5.056$, $(Y_1)+(Z_1) \leq 5$. But if $(Y_1)+(Z_1) = 5$, then $W=7$ which in light of [21], is also impossible.

If $(Y_1)+(Z_1) = 4$, then $W=132$ which in light of [21], is also impossible.

If $(Y_1)+(Z_1) = 3$, then $W=257$ which in light of [21], is also impossible.

If $(Y_1)+(Z_1) = 2$, then $W=382$ which in light of [21], is also impossible.

Hence $(X_1) \neq 1$.

Suppose $(X_1) = 3$. Then from [3] and [4] we get

$$(Y)(Z)(W) = (3)(2^6)(5^6) \quad [23] \text{ and}$$

$$Y+Z+W = (2)(3)(79) \quad [24].$$

Once again 5^3 divides one of Y , Z , or W but 5^4 does not divide any of them, so without loss of generality we may assume $Y=125(Y_1)$ and $Z=125(Z_1)$. Hence [23] and [24] become

$$(Y_1)(Z_1)(W) = (3)(2^6) \quad [25] \text{ and}$$

$$125(Y_1)+125(Z_1)+W = (2)(3)(79) \quad [26]$$

Since $\frac{(2)(3)(79)}{125} = 3.792$, we see that $(Y_1)+(Z_1) \leq 3$. If $(Y_1)+(Z_1) = 3$, then

[26] says $W=99$ which contradicts [25]. If $(Y_1)+(Z_1) = 2$, then [26] says $W = 224 = (32)(7)$, which contradicts [25].

Thus, $(X_1) \neq 3$.

Finally we have shown that $(X_1)=4$ and [3] and [4] become

$$(Y)(Z)(W) = (3^2)(2^4)(5^6) \quad [27] \text{ and}$$

$$Y+Z+W = (5)(79). \quad [28]$$

Since $5^4 > (5)(79)$, it is clear that 5^4 does not divide any of Y , Z , or W . Next 5^3 cannot divide exactly 2 of Y , Z , or W , otherwise from [28] we would see that 5 would divide the remaining one contradicting [27]. Thus 5^3 divides one of Y , Z , or W , 5^2 divides one of the remaining two, and 5 divides the third. Without loss of generality we obtain $Y=125(Y_1)$, $Z=25(Z_1)$, and $W=5(W_1)$. Thus [27] and [28] become

$$(Y_1)(Z_1)(W_1) = (3^2)(2^4) \quad [29] \text{ and}$$

$$25(Y_1)+5(Z_1)+ (W_1) = 79 \quad [30]$$

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1		9	45	165	495	1287	3003	6435	12870
1	8	36	120	330	792	1716	3432		6435
1	7	28	84	210	462	924	1716		3003
1	6	21	56	126	252	462	792		1287
1	5	15	35	70	126	210	330		495
1	4	10	20	35	56	84	120		165
1	3	6	10	15	21	28	36		45
1	2	3	4	5	6	7	8		9
		1	1	1	1	1	1	1	1

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Your job is to write numerals on pages in a book. You are paid by the number of digits you write. You wrote 642 digits. How many pages did you number?

Solution: Assume no Roman numerals at the front of the book.

Digits 1 through 9 take 9 pages.

10 through 99 take another 180 digits.

A total of 189 digits have been used. $642 - 189 = 453$.

$$\frac{453}{3} = 151.$$

$$9 + 90 + 151 = 250 \text{ pages.}$$

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Resources

Brumbaugh, D. K., Ortiz, E., Gresham, G. (2006). *Teaching Middle School Mathematics*. Mahwah, NJ: Lawrence Erlbaum Associates.

Brumbaugh, D., Rock, D. (2001). *Scratch Your Brain C1*. Pacific Grove, CA: Critical Thinking Books and Software.