

Problem Solving

As you go through the notes for this workshop, it will be advantageous for you to work the problems discussed before reading the presentation. That will sharpen your problem-solving skills.

Recent years have brought renewed emphasis on problem-solving. Many claim that problem-solving has been an essential ingredient since the beginning of formalized consideration of mathematics. Textbooks dealing with the history of mathematics frequently contain statements that mathematical development should be in the context of problems. Newman (1956) wrote that it is generally accepted that the ancient Egyptians invented geometry (geo meaning Earth, metry meaning measure) in order to restore land boundaries associated with landmarks swept away periodically by flooding of the Nile River.

Mathematical history provides a wealth of information about problem-solving and can be used as a rich source of ideas to attract the attention of students, integrate mathematics with history, provide connections between other disciplines, show applications, link concepts within the mathematical curriculum, and provide challenges for individuals. Historical examples come from Descartes and Euler, both of whom contributed to the development of topology. The “Seven Bridges of Konigsberg,” which Euler proved impossible, is considered to be one of the foundation stones of topology, according to Newman (1956).

Here is a problem to solve that leads to another historic consideration. You are giving a party. You are not a guest.

First doorbell ring brings one guest.

Second doorbell ring brings 3 new guests.

Third doorbell ring brings 5 new guests.

Etc. Each new doorbell ring brings 2 more new guests than last one.

How many guests enter on the 20th ring?

39. The numbering is always one less than two times the doorbell ring number. The first one is $(2)(1) - 1 = 1$. The second is $(2)(2) - 1 = 3$. The third is $(2)(3) - 1 = 5$. And so on.

What is the grand total number of guests after the 20th ring?

400. After the first ring there is a total of 1 person. After the second ring, the total is 4 $(1+3)$ people. After the third the total is 9 people $(1+3+5)$. There is a pattern, 1, 4, 9, . . . , perfect squares.

Are generalizations possible? Yes

Sum of first n consecutive odd counting numbers = n^2 gives the total number of people at the party.

$$\text{Sum of first } n \text{ consecutive counting numbers} = \frac{n(n + 1)}{2}$$

gives the number of people entering on any given ring. Note that this generalization is similar to the one for the next historical problem.

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Gauss, who, as an elementary student, quickly found the sum of the first 100 consecutive counting numbers to be 5,050 by using something similar to the following process.

$$\begin{array}{cccccccc} 1 + & 2 + & 3 + \dots + & 98 + & 99 + & 100 \\ \underline{100} + & \underline{99} + & \underline{98} + \dots + & \underline{3} + & \underline{2} + & \underline{1} \\ 101 + & 101 + & 101 + \dots + & 101 + & 101 + & 101 \end{array}$$

Because there are one hundred “101s”, the sum would be $(100)(101)$. But this sum is twice what it should be because each addend was used twice. The sum of the first one hundred consecutive counting numbers is $\frac{(100)(101)}{2}$. In general,

the sum of the first n consecutive counting numbers can be found by using $\frac{(n)(n - 1)}{2}$.

WHAT IS PROBLEM-SOLVING?

We are about to discuss problem solving, how to do it, and how to involve students in it. But what is problem solving? Before we can do it, talk about it, or teach it, we should discuss what problem solving is, and what it is not. Problem solving is a process that evolves through life. Problem solvers encounter situations that intrigue them enough to work through a mystery to arrive at a satisfactory solution. Problem solving makes use of previously acquired knowledge, skills, and comprehension, which are then synthesized into a new format that provides avenues to resolve the question at hand. The expectation is that problem solving is going to require the student to use acquired facts and information in the problem to solve the mathematical mystery in which they are currently engaged. Sometimes a new method approach, or concept is created as one deals with a problem-solving situation. Most people think problem solving can be taught. Some people think problem solving evolves out of the practice of solving problems. There is more involved in defining problem solving, but this should suffice to get discussion, and thought, started.

NCTM lists “Mathematics as Problem-solving” as Standard 1 in each of the K–4, 5–8, and 9–12 Curriculum Standards in Curriculum and Evaluation Standards for School Mathematics (NCTM, 1989b). Standards 2000 (Principles and Standards for School Mathematics, NCTM, 2000), NCTM’s revision, consolidation, and extension of Standards, Professional Standards, and Assessment Standards, continue the emphasis on problem-solving. Close investigation reveals that problem-solving is intertwined throughout the curriculum.

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REAL-WORLD PROBLEMS

Discussion within the NCTM Standards deals with the idea that mathematics students need problem-solving problems from their world. The Standards (NCTM) say that, as students mature mathematically, problem situations spring from within the study of mathematics. Problem-solving questions can resolve dilemmas from the students' real world and from their world of mathematical growth, as was often the case for ancient mathematicians. It should be noted here that the Standards and the Addenda Series (NCTM, 1989a) contain a variety of settings that could be used to develop problem-solving skills with students.

PROBLEM-SOLVING TODAY

It is difficult to discuss problem-solving without giving respect to George Polya. Many consider Polya to be the father of modern thought on problem-solving. In 1945 Polya wrote *How to Solve It*, which provides a wealth of information and includes a list of four problem-solving steps, which are:

Understand the problem.

Make a plan for solving the problem based on data and ideas given.

Carry out the plan.

Look back at the solution. (Polya, 1945/1973)

Comprehension, planning, implementation, and follow-up are basic steps involved in the business world. The similarities between these and Polya's list could be additional selling points for students. Investigation of the literature on problem-solving will show a variety of lists of steps but, in almost every case, Polya's four steps form a basic framework. Those four steps are generic problem-solving skills that can be applied in a multitude of real-life settings.

LEARNING PROBLEM-SOLVING

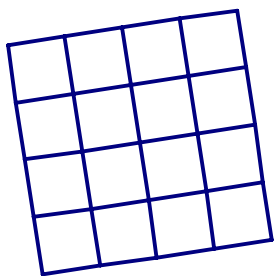
How does one learn to solve problems? There is a limit to how long we should talk about a subject. At some point discussion should motivate us to take action. It will be difficult to convince students to become problem solvers if you are not a problem solver yourself. This does not mean that you must solve every problem you encounter. Some problems will appeal to you and some will not. You do need to model the desired behavior. As you solve problems, you should experience a joy and excitement that can be transmitted to your students. This enthusiasm helps students want to replicate your actions and acquire at least a part of the thrill you project about resolving a situation.

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ONLY ONE ANSWER

A good beginning point is investigating what constitutes a problem-solving problem. Consider the following: “Four CDs are purchased at \$13.98 each. How much money was spent?” For the time being, ignore sales tax, volume discounts, whether this is a store or buying club price, and so forth. At some point, to add realism to the situation and to entice the students to become involved in the problem-solving process, we do need to insert such options into our discussions or students may discount the whole process as unrealistic and unappealing because of the overlooked items. The whole point of problem solving could be missed because of our failure to attend to details that students might view as significant in real-world settings. Remember, we are attempting to convince students to become problem solvers. To that end, we must present situations and questions that will attract their interest. Once we have them focused on the process necessary to solve problems, we can shift their efforts to other areas that may not have real-world applications. The emphasis will eventually shift from the problem itself to the method and procedures needed to answer a question.

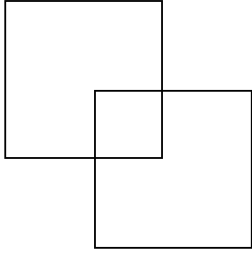
“Four CDs are purchased at \$13.98 each. How much was spent?” Is this a problem-solving problem? Certainly there are words involved and, to many, that indicates problem solving. Unquestionably, the words need to be translated into symbols, numbers, and expressed as an equation that will lead to the all-important answer. Many mathematics educators now think that, although the answer is important, it is not necessarily the “end all, be all” of the process. In many instances, there is no right answer. For example, “How many squares do you see?”



Can the number of squares you see be incorrect? Will the number of squares you see be the same as the number of squares another individual sees? What happens if a student finds more squares than you do? We can present situations in which there is more than one correct answer.

How about this one? How many squares do you see?

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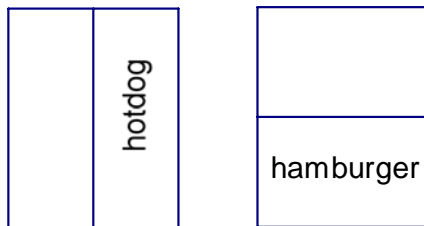


Most people will say 2 or 3. Sometimes the answer is 6, no, 7. The 6 come from inserting the imaginary segments that would create a cube. That answer is followed by 7, which comes from the “overlapping” little square. One student provided a most interesting perspective by announcing there is only one square in the figure. The explanation involved erasing the little square from the figure and then the outline would not be a square.

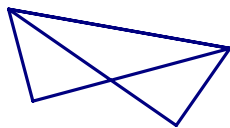
Another example involving more than one right answer can be found in statistics. Suppose a class completes a survey in which the birth month of each student is listed. The objective is to graph the data and discuss the mode. For this example, assume the number of births is equal in two different months. The graph would be bimodal, which can confuse some students. They are accustomed to seeing one right answer, especially when they think the mode will be the month with the greatest number of births. Even though they have learned that a bimodal result is acceptable, their prior conditioning for one and only one right answer may cause them to resist.

Recall the paper folding exercise from page 2 of the Skills in Teaching Mathematics Workshop text. That is another example where there is no one right answer. Different answers were derived for the same question depending on the procedures followed.

Consider how a sheet of paper can be folded in half. Hamburg and hotdog are the two most common ways.

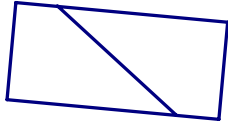


People will say they know they have folded the paper in half because the parts are the same size, they have the same area, and that the edges match. Are there other ways a rectangular piece of paper can be folded in half? Yes. For example,



, but the problem is that the edges do not match. Yet, cut along the diagonal and rotate one of the triangles 180 degrees, the edges of the two triangles will match. Thus, how one can tell if a paper is folded in half needs to be altered. Are there other ways? Yes. For example, in this one,

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the endpoints of the diagonal are equidistant from the vertices closest to them, respectively. Similar situations could be used, essentially giving an infinite number of ways to divide a rectangle in half.

WHAT IS A PROBLEM-SOLVING PROBLEM?

“Four CDs are purchased at \$13.98 each. How much was spent?” We have now determined that a problem may have more than one right answer, and that there is a multitude of acceptable methods for determining that answer. However, is this CD question a problem-solving problem? Some perspectives and guidelines are needed to respond. Krulick and Rudnick (1987) gave a list of four essentials to determining if a situation is a problem-solving problem:

- A non-routine solution is necessary
- A challenge is presented
- An individual accepts the challenge
- A positive attitude about problem-solving is being fostered

“Four CDs are purchased at \$13.98 each. How much was spent?” The idea of a routine solution gives a clue about determining if this is a problem-solving problem. Undoubtedly you have tackled similar problems where these numbers or descriptors were different, but essentially the idea was the same. This becomes a routine problem for you and would not be classified as a problem-solving activity.

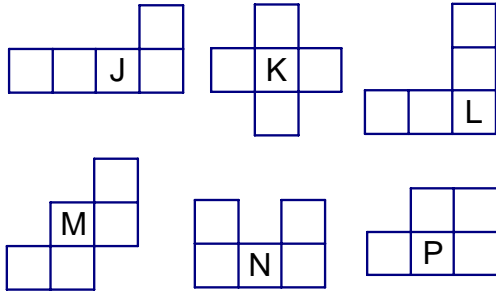
On the other hand, the four-CD problem would present a huge dilemma to many students. A first- or second-grader might be dealing with buying three pencils at the school store when each one costs 15¢. Reflexively, we would probably multiply the 15¢ by three and determine we need 45¢ (sales tax?). The difficulty is, first- and second-graders typically do not know how to multiply. Some of the more resourceful ones will perhaps add, but that is not very likely, particularly with the first-graders. On top of the addition skills, if the value is written \$0.15, it must be remembered that these children will not have encountered decimals at a level that would permit them to deal with this situation. The possibility does exist that coins could be used, particularly in the case of second-graders, to arrive at the solution. Finally, the student could avoid the issue by going to the store, presenting a dollar bill, and hoping that it will cover the cost of the three pencils. If it will not, the “clerk” will inform the student of the shortage and the additional amount needed. Certainly at this level, the customer must have a significant amount of trust in the clerk.

“Four CDs are purchased at \$13.98 each. How much was spent?” It appears that this may or may not be a problem-solving problem. It depends on the mathematical maturity of the individual of whom the question is asked. The answer is also influenced by whether or not the person cares enough to pursue

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the solution - - a challenge is offered and accepted. Finally, when the problem has been solved, the individual should feel good about the accomplishment, which begins to build a positive attitude about problem solving.

As problems are solved, ideas are bantered about either with others or in one's own mind. The discussion about folding a sheet of paper would be an example. The ability to generate ideas is important to the overall process of becoming a problem solver. Suppose a class is given this set of shapes, which are all comprised of congruent unit squares. The teacher is thinking of a particular



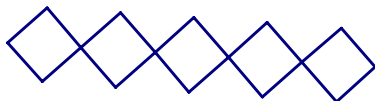
question regarding the shapes, and the students' task is to determine the question the teacher has in mind. Several different questions will be generated, and each of them can be used to stimulate class discussion. Some examples and possible associated discussions follow.

“What is each shape's area?” Because each figure is comprised of five unit squares, the realization that all areas are equal comes rather quickly. Area was not the question in mind, yet that discussion provided the opportunity to investigate some mathematical concepts that may be valuable for the students.

“What is each shape's perimeter?” This usually follows the area question, which opens the door to some interesting discussion. If the figures are considered in the order they are alphabetized, the perimeter is 12 units for each of J through N. Students generally conclude that all six shapes have the same perimeter. What a wonderful opportunity to discuss jumping to conclusions before adequate investigation is conducted. Perimeter was not the original question.

“What is the minimum perimeter of a figure comprised of 5 unit squares?” Ten is the minimum because the least number of sides are exposed, as shown in configuration P. Someone might suggest stacking one square on top of another, thus decreasing the perimeter. Good thinking (and that should be noted), but it is outside the rules of a plane figure. Minimum perimeter was not the question either.

“What is the maximum perimeter of a figure comprised of 5 unit squares?”

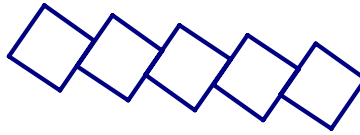


Twenty units since all edges are exposed.

Discussion might bring objections from students because they thought sides had

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to be adjoining. This presents the opportunity to review self-imposed restrictions and assumptions. It also provides the opportunity to discuss situations where half of each touching edge is shared. That means the other half-length is exposed.



What is the perimeter of the figure?

What if the overlap takes three fourths of a side length and the remaining one fourth is exposed? Is this perimeter equivalent to the perimeter for a one-half overlap? Why or why not? And so on. However, that was not the question either.

“Can these pieces be put together to form a square?” If the group has worked with tessellations or tangrams

(<http://enchantedmind.com/puzzles/tangram/tangram.html>), this response appears quickly. Once it is concluded that the pieces cannot be used to form a square, “Why?” becomes a useful question to ask. The six shapes (J, K, L, M, N, and P) are made up of unit squares with a total area of 30 square units. The pieces, when put together, would have to give a square of a counting number side length, so a total of 36 unit squares would be necessary to even begin to perform the desired task.

The intended question was “Which of these shapes can be folded to form an open box?” This solution involves flexible thinking and overriding self-imposed restrictions. The explanation of which shapes can and cannot be folded to form the open box may provide you with clues about the abstract thinking abilities of individuals. Most students will be able to visualize some solutions. Other shapes are much more difficult to imagine being folded into an open box. The students may need to make models and actually perform the folding. This should be encouraged and is an excellent way to meet the needs of different ability levels or learning modalities. The emphasis should be on arriving at the conclusion through some “legal” or acceptable manner, not whether or not the most abstract or efficient method was used.

There is a real world connection between the open box question and a classic calculus question relating to cutting corners out of a rectangular sheet to create a box of maximum volume while considering minimum product consumption and waste. If the student is unable to visualize and think of the flat pattern being folded to form a box, there might be limitations on that student’s ability to answer the question. In the calculus problem, once the box is formed, the student is expected to manipulate the dimensions to provide the desired response. We would assume that a student who is able to visualize the anticipated result should have a greater chance of understanding the total concept.

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For an interactive version of this, go to

<http://www.cut-the-knot.org/Curriculum/Calculus/BoxVolume.shtml>.

There is a significant difference between “Four CDs are purchased at \$13.98 each. How much was spent?” and “Consider the set of six shapes and determine the question.” The second question would be a problem-solving situation for almost anyone, as the calculus problem. On the other hand, the price of four CDs would not be a problem-solving situation for many, depending on mathematical maturity.

UNEXPECTED OPPORTUNITIES

The next problem appears frequently in the literature, can be solved by students of multiple ability levels, and can be pursued to greater depths. You are given a double-pan balance capable of holding as many objects on either side as



desired.

You are also given a set of eight blocks and it is known that one of the blocks is lighter than the other seven, which all weigh the same. It is impossible to tell, by looking or lifting, which block is the light one. The balance is used to determine the answer. What is the fewest possible number of weighings necessary to guarantee identification of the light block each time the problem is done?

The typical answer is three weighings, which will work, but it is not the fewest. A person could randomly select two blocks, placing one on each pan of the balance. If the pans do not balance, the lightweight would be identified, but that would be luck and could not be depended on time after time, which is one of the requirements of this problem.

When confronted with a difficult situation, mathematicians frequently try to solve an easier problem and then apply the lessons learned to the more complex issue. Simplifying this problem, consider only two blocks and identify the lightweight. Placing a block on either side of the balance will show the desired result in one weighing. For some students, it might be necessary to demonstrate that the result is the same if the blocks are placed on the other side of the pan balance.

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With three blocks, some students may not understand that placing two blocks on one pan and one on the other will always give an unbalanced result. That should be resolved. Blocks not currently in use should not be overlooked. Some students may not realize that with the light one identified, there is no need to weigh any blocks not in use. For example, suppose one block is on either side and the lightweight is on the right pan. That tells two things, one of which is not obvious to some. The block on the left pan is heavy and most students will grasp that idea. However, some students will ask to use the other block from the table because it might change the results. Remember that there is only one light block in the set. It has been identified. All the other blocks weigh the same. This is an important idea and it must be considered throughout the solution of this problem. Therefore, it is imperative that you spend time resolving this issue for all students at this beginning level.

Four blocks require how many weighings? Two are placed on each pan and one side will hold the lightweight. Once the lightweight is determined as being in one of the two pairs, the solution is complete because the situation with one block on each side has already been determined to take one weighing, giving a total of two weighings to solve the four block problem. If the blocks do not balance when placing one block on each side of the balance, the lightweight is identified in one weighing, but that is luck. When placing one block on each side of the balance, if the blocks do balance, weighing the second pair will identify the light block but that takes two weighings. Students need to realize the advantages of placing the optimal number of blocks on each side, so the idea of two on each side is more beneficial in the long run.

Five blocks can be solved just like four, leaving one block out. If the four balance (two on each side), the lightweight is identified in one weighing, but that is luck. If the four do not balance, it is important that the students realize the excluded one is not light and would be exempt from the weighing process. Again, this abstraction is difficult for some students to comprehend and care must be taken to assure their understanding. It should be noted that this is another application like the cases considered when working with three blocks.

Six blocks can be done two different ways. Typically, students will want to place two blocks on each pan. If they balance, the lightweight is one of the two not being used. If the four on the scales do not balance, then the two not being used are heavy and only one more weighing is needed to identify the lightweight. Either way, the lightweight is identified with two weighings.

The second way of working with this problem has three blocks placed on each pan. The lightweight will be identified as one from a set of three. The three problem has already been done in one weighing. The problem is solved again in two weighings. The placing of three on each side extends the idea of optimal selection.

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Seven blocks are treated like five were.

Eight blocks, like six, can be done more than one way. Students typically select four blocks on each side of the scales. In this setting, the lightweight is identified as one of four, which can be resolved with two additional weighings. This approach gives a total of three weighings.

The other way for doing eight blocks involves selecting three blocks for each side of the scale. If the three on each side balance, the lightweight must be one of the two blocks not being used, because the six being considered all weigh the same. One more weighing will solve the problem and a total of two weighings is required. If three on each side do not balance, the two not in use are now known to be heavy along with three on one side of the scales. However, that means the lightweight is one of three blocks, a situation that can be solved with one more weighing. Either way the problem can be solved with two weighings.

This problem can be extended to show patterning and algebra. Nine blocks can be done in two weighings. Put three on each side and if they balance, it is known that the remaining three contain the lightweight and require only one more weighing for identification. If the three on each side do not balance, it is known that the remaining three are all heavy and one more weighing, using the side with the light three, identifies the lightweight.

Ten blocks require three weighings. The 10 blocks can be placed 5 on a side and then the lightweight will be in one of two sets of 5, a situation that requires two additional weighings to resolve. An initial setup of 4 on a side or 3 on a side will also lead to three needed weighings (excluding luck).

Certainly, this process could be continued using the pan balance and, for some students, that is appropriate. For some students it is time to shift to a more abstract approach. Ask how many blocks can be done with three weighings. Once this answer is determined (27), along with an appropriate explanation (perhaps accompanied by some demonstrations - - it takes 9 blocks on each pan and that is one weighing. It is known that 9 blocks can be solved in two more weighings.) of why the given response is correct, ask how many blocks can be handled with four weighings. Better problem solvers will accept this as a clue that there must be a generalization, if they have not already started searching for it. The initial problem of weighing blocks to identify the lightweight has opened avenues for a variety of student abilities. Four weighings will handle up to 81 blocks (27 on each side, knowing 27 can be done in three weighings).

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Now the pattern begins to emerge.

3 blocks require one weighing

9 blocks require 2 weighings

27 blocks require 3 weighings

81 blocks require 4 weighings

3, 9, 27, 81 or 3^1 , 3^2 , 3^3 , 3^4 , or, in general, 3^x , and you have an algebraic insert at the end of the pattern.

BECOMING A PROBLEM SOLVER

You should have been working the problems presented so far. If you have, you are progressing toward becoming a problem solver. If you are going to expect your students to be problem solvers, you must model appropriate behavior. Trying the problems presented is a great beginning. If you do not accept the challenges of these problems, how can you, in good conscience, expect your students to attempt the problems you pose for them? It becomes a matter of practicing what you preach. One beauty of becoming a problem solver is that your excitement about learning will spill over into other areas of teaching. Then, both you and your students will benefit.

TEACHING PROBLEM-SOLVING

Each teacher must do problem solving. In the process of doing, appropriate problem-solving behavior is modeled. How did we learn to solve problems? In most cases, the ability has taken years to evolve, and there was little, if any, conscious awareness of its development. Now the expectation is that you teach this ability to students. How can students be taught to become problem solvers without our modeling the desired result?

As problem-solving skills are developed, several issues warrant consideration. Specialized problem types create difficulties: number problems, rate problems, age and coin problems, percent problems, mixture problems, and so on. The trouble is that if the first week is spent doing age problems and then percent problems are done the second week, at the end of the second week the students typically have forgotten how to do the age problems.

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The problems presented must be appropriate for all ability levels and, yet, fit into the curricular topics being covered. The block problem fits a variety of need levels:

Concrete students can solve the problem (with patience)

Patterning leads to generalization for better students

Generalization leads to algebra for top students

Presenting problems appropriate for students who have different ability levels is not overly difficult, thanks to the multitude of resources now available.

<http://www.olemiss.edu/mathed/contest/contests.htm>

www.umassd.edu/mathcontest/

<http://www.whitehouse.gov/kids/math/index.html>

<http://mathres.kevius.com/problem.html>

<http://mathforum.org/>

Polya's How To Solve It

NCTM yearbook on Problem-solving

Krulick & Rudnick -- Problem-solving

Mathematics Teacher (NCTM)

Teaching Middle School Mathematics (NCTM)

Teaching Children Mathematics (NCTM)

Texts have Problem-solving information in TEs

<http://enchantedmind.com/puzzles/tangram/tangram.html>

However, using problems appropriate for the curriculum being presented is not always that easy; planning is crucial. One factor that should be considered is how frequently problem solving should be used. The easy answer is "As often as possible," but that response does not provide much information. Problem solving can be frustrating, tedious, and tiring, as well as invigorating and stimulating. The frequency of presentation requires a delicate balance. Part of the question can be resolved by establishing a program whereby a problem-solving exercise is presented on a regular basis. This problem may or may not be related to the curriculum being covered. As problems are offered, time must be allowed for discussion of solutions. A possible reaction of some students is not to accept the challenge of the problem and, therefore, they will not be interested in the solution.

You need to incorporate problems appropriate for varying student ability levels, gathered from different subject areas, exemplifying a variety of real-world applications, and designed to reach a multitude of objectives. Many of these problems do not need to be invented. They do exist in resources like those listed earlier in this chapter.

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THINKING GAMES AND PUZZLES

A method of stimulating creative growth that has shown promise for many years is the use of thinking games. This can be something as common as tic-tac-toe. “Standard” rules could be followed or the guidelines could be reversed so that three in a row, column, or diagonal would lose rather than win. This seemingly minor change creates a setting in which thought processes must be altered significantly if an individual is to win. This is a formative step in becoming a flexible thinker. Games like chess, checkers, or Chinese checkers, card playing, billiards, and so forth, can be played with a premium placed on explanation of strategies used.

Puzzles (Tower of Hanoi, Adam’s Cube, Sneaky Squares, and so forth) provide rich opportunities for the development of problem-solving skills. The Tower of Hanoi, offers the opportunity to develop thinking patterns, strategies, generalizations, and algorithms. The classic question asked involves determining the minimal number of moves required to get a stack of progressively smaller disks from one pin to another, moving only one disk at a time, and never placing a larger disk on a smaller one. NCTM provides a collection of free interactive tools and activities at <http://illuminations.nctm.org/tools/index.aspx>. One of the tools is found at http://illuminations.nctm.org/tools/tool_detail.aspx?id=40 and provides the opportunity to work the Tower puzzle electronically.

There is a folklore tale about a sect of monks who have a Tower puzzle consisting of three pins and 64 disks. They are able to correctly move a disk every second and work in shifts so the operation is nonstop. Their belief is that when they complete the task, the world will end. The question coming out of the tale is to determine how long it will take the monks to complete their task.

Number Of Disks	Number Of Moves	Rewriting Exponents	Generalization
1	1	$2 - 1$	$2^1 - 1$
2	3	$4 - 1$	$2^2 - 1$
3	7	$8 - 1$	$2^3 - 1$
4	15	$16 - 1$	$2^4 - 1$
	•		
	•		
	•		
n			$2^n - 1$

The monks are going to need $(2^{64}) - 1$, or 18,446,744,073,709,551,615 seconds. Dividing that number of seconds by 3,600 seconds in an hour, 24 hours in a day, and 365.25 days in a year gives 5.845×10^{11} or 584,542,076,576 years. What we want the students to do is develop a strategy, and extend their efforts enough to see the advantage of knowing and using mathematics to answer questions. If

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the student elects not to accept the challenge offered, there is no need for the generalization.

DEVELOPING GOOD PROBLEM SOLVERS

There are several traits held by problem solvers. No good problem solver will exhibit all these traits, and the list given here is not intended to be exhaustive, but it is indicative of the thinking process of many problem solvers. A good problem solver will show:

- Thinking
- Reasoning
- Hypothesizing
- Educated guesses with follow-up
- Idea getting
- Patience
- Persistence
- Pattern searches
- Generalization building
- Flexible thought
- Drawing on ideas from others
- Inspiring others to begin thinking
- Creative suggestions
- Stating the obvious
- Skipping steps
- Talking to self

Do not overlook students who converse with themselves as they work through a problem. That person knows what questions to ask and answer. Most of us have had a conversation with ourselves and, in the process, determined some insights that lead to an appropriate conclusion.

Often a good problem solver will appear to skip steps. These steps may go unsaid because they are trivially obvious to the solver. The steps are being done mentally. The skipping is acceptable, as long as the solver can communicate the solution, including the skipped steps if they are necessary for another individual. The individual who skips steps is frequently identified as a good problem solver.

Problem Solving

DEVELOPING PROBLEM-SOLVING SKILLS

Suppose you were attending a party. The person giving the party is asked to give the ages of children. The following information is given to the requesting individual:

There are three children.

Each child has a counting-number age.

The product of their ages is 72.

The sum of their ages is the same as this house number.

The guest thinks about the information, goes outside, looks at the house number, goes back inside and says, "I need more information." At that time, the guest is told, "The oldest likes strawberry ice cream." On hearing the statement, the guest gives the correct ages of the children.

Child A's Age	Child B's Age	Child C's Age	Sum of Ages
72	1	1	74
36	2	1	39
24	3	1	28
18	4	1	23
12	6	1	19
9	8	1	18
18	2	2	22
6	6	2	14
12	3	2	17
9	4	2	15
8	3	3	14
6	4	3	13

The table shows the solution to "strawberry ice cream." After seeing the house number, the guest realizes there is a need for additional information. If the guest had seen a house number of 74, the ages had to be 72, 1, and 1. This particular combination of ages bothers many students, and yet, it is at least mathematically possible. If the guest had seen a house number of 39, the ages would be known to be 36, 2, and 1. If the guest had seen a house number of 28, the children would be 24, 3, and 1. The only possible house number that demands more information is 14. Because there are two sets of ages yielding a house number of 14, the guest needs to know the oldest. The children are counting number ages only, so twins are considered to be the same age. In the case of the ages 6, 6, and 2, there would be no clear oldest. With ages of 8, 3, and 3, the 8-year-old is clearly the oldest. Strawberry ice cream has nothing to do with the solution other than it provides the opportunity to have a statement about the oldest child. A nice serendipity related to this problem is that many people conclude the children must be 9, 8, and 1 because they count 18 letters in "strawberry ice cream."

Buffon's principle, developed by the Comte de Buffon in 1760 (Eves, 1967, p. 94), is a short class activity that assists in developing communication and

Problem Solving

reasoning skills. Each student needs a handful of toothpicks (20 to 30), a blank sheet of paper, a pencil, and a straight edge. Students should draw parallel line segments on the paper a toothpick's length apart. It is crucial that the segments are a toothpick length apart. Next, have each student hold the toothpicks above the center of the paper and drop them (the drop should be a minimum of 2 to 3 feet to create a valid spread). A toothpick crossing a drawn segment is an "ON" toothpick and one not crossing the line segment is an "OFF" toothpick. Any toothpick off the paper, or hanging over the edge, is not considered. The activity is simulated at <http://www.mste.uiuc.edu/reese/buffon/buffon.html>.

Assuming the students do the activity with toothpicks, one by one, students call out their number of ONs and OFFs, which added to the respective total. Compute $\frac{(2)(\text{ONs})}{\text{OFFs}}$ after each new total is added. The students discover that as the number of trials increases, the average of twice the number of toothpicks ON divided by the number of toothpicks OFF approaches π . This summary is particularly nice if done on a spreadsheet. Similar activities using coat-hanger wire and a square tile floor can be developed. The floor-tile seams would be the parallels and the coat wire would be cut equal to the length of the tile side (representing the toothpicks). This is another example of an activity that could be inserted at a variety of points throughout the curriculum, depending on the ability and background of the students.

As problem-solving skills are developed, care must be taken to ensure problem-solving and new content are not taught at the same time. Problem-solving problems can be used to generate a need for new content. New content might provide the background for an exciting problem. Teaching the two new things at once may lead to confusion of the students and should be avoided. There will be times when, through coincidence, it will happen that the two will appear at the same time, but it should not be planned that way.

Consider the typical introduction of a new concept.

Give a formula.

Do a few examples.

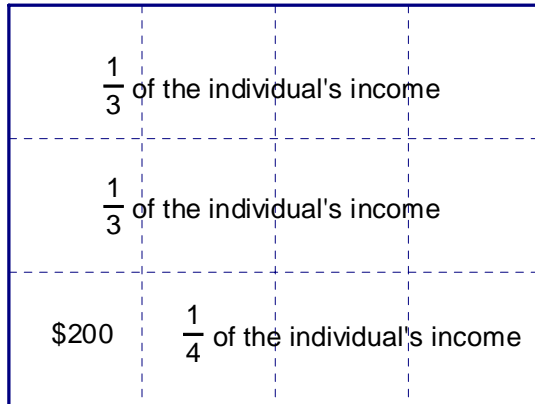
Practice

Problems all similar with different numbers

How much better it would be if, rather than assigning 20 similar problems, only five were given, with the expectation that each one be solved more than one way. Suppose one-third of an individual's income is spent on housing, another third is spent on transportation and education, and 25% on food and entertainment. The individual has \$200 left for saving, giving, investing, and shopping? How much does the individual make a month? Algebraically, a solution could be determined through the equation $\frac{x}{3} + \frac{x}{3} + \frac{x}{4} = x - \200 ,

Problem Solving

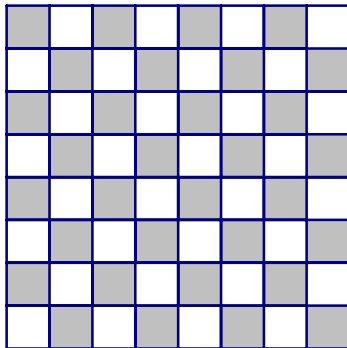
where x represents the monthly income. Solving, $\$200 = \frac{x}{12}$, which gives that $x = \$2400$.



Geometrically, the total income (big rectangle), is divided into thirds horizontally, and fourths vertically. The net result is a total of 12 congruent squares, four of which represent a third and three of which represent a fourth. There is one square left after the fourth and both thirds are represented. But, we know that the left over money is \$200. Since there are 12 congruent squares, the total income must be $12 \times \$200 = \2400 . The problem is solved a second way and the versatility of the student should be enhanced.

SOME PROBLEMS TO DO

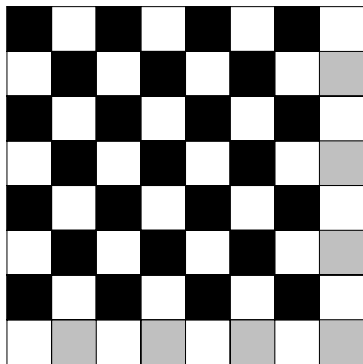
How many squares are on a checkerboard?



The typical response is that there are 64 squares, often followed by a pause, and a revised response of 65 squares. Sometimes, after another pause, there is a proclamation that there are more than 65 squares, but there is uncertainty on how to proceed with the solution.

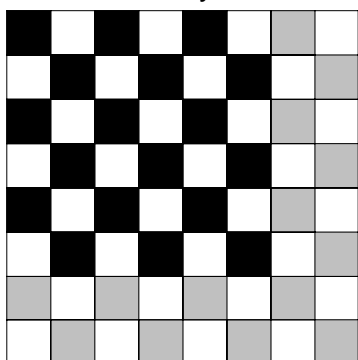
The easiest way to solve the problem is to establish a pattern, starting with the large square, which is 8 units by 8 units. The next smaller sized square is 7 units by 7 units, and there are four of them.

Problem Solving



The seven by 7 square is shown in black and white, superimposed on the 8 by 8 square. The 7 by 7 square can be moved to the right one unit, thus giving a different 7 by 7 square. Then it can be dropped down one unit, giving an third 7 by 7 square, and finally moved to the left to give the fourth 7 by 7 square.

Similarly, there are nine 6 by 6 squares on the checkerboard.

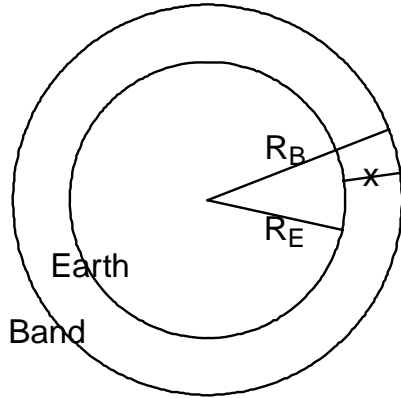


The pattern is extended to have sixty-four 1 by 1 squares, giving a grand total of $1+4+9+16+25+36+49+64 = 204$ squares on an 8 by 8 checkerboard.

Generalize the original question to an “N by N” checkerboard. What happens if the checkerboard is N by M where $M \neq N$?

Suppose the earth is a sphere and the circumference at the equator is 25,000 miles. A band is placed around the earth, concentric with the equator. The circumference of the band is 25,000 miles + 10 feet. A standard sheet of 8.5 - by 11-inch notebook paper is about 0.003 inches thick. Would that paper fit between the band and the equator in any of its three configurations (thickness, width, length)? What is the thickest thing that could fit between the band and the equator?

Problem Solving



R_E = Radius of the Earth

R_B = Radius of the Band

$x = R_B - R_E$

C_E = Circumference of the Earth

C_B = Circumference of the Band

$C_E = 25,000$ miles

$C_B = 2\pi R_B$

$C_B = 2\pi(R_E + x)$

$C_B = 2\pi R_E + 2\pi x$

$C_B = C_E + 10$ feet

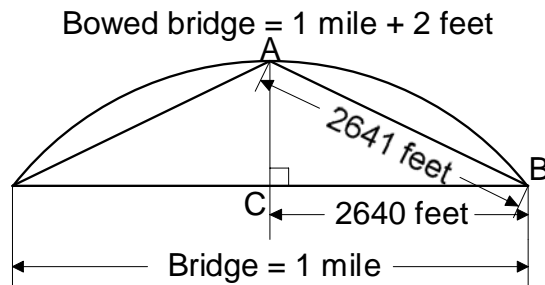
$C_B = 2\pi R_E + 10$ feet

So, $2\pi R_E + 2\pi x = 2\pi R_E + 10$ feet

$2\pi x = 10$ feet Subtracting $2\pi R_E$ from both sides

$x = \frac{5 \text{ feet}}{\pi} \approx 1.59$ feet

A mile-long, horizontal bridge is built with no expansion joints. Neither end will move. The bridge expands to a length of 1 mile + 2 feet due to temperature changes. The expansion causes the bridge to bow up in the middle. What is the distance between the center of the bridge in its normal and in its expanded state? The value of this question is suspect because, in the real world, bridge design must allow for expansion and contraction. However, the answer is so surprising that it is worthy of consideration.



Problem Solving

This figure is not drawn to scale. In actuality, segment AB and the arc above it are almost collinear. ABC is a right triangle, so the Pythagorean theorem can be used to solve for the height of point A above the center of the flat bridge.

$$AC^2 = (2641 \text{ feet})^2 - (2640 \text{ feet})^2$$

$$AC = \sqrt{(2641 \text{ feet})^2 - (2640 \text{ feet})^2}$$

$$AC = \sqrt{(5281 \text{ feet})^2}$$

$$AC \approx 72.67 \text{ feet}$$

A school has a hall with 1,000 lockers, all of which are closed. A thousand students start down the hall. The first student opens every locker. The second student closes all lockers that are multiples of two. The third student changes (closes an open locker or opens a closed one) all multiples of three. The fourth student changes all multiples of four. And so on. After all students have entered the school, how many lockers are closed and which ones?

Look at Locker # 24

Kid 1 opens; Kid 2 closes; Kid 3 opens; Kid 4 closes; Kid 6 opens; Kid 8 closes; Kid 12 opens; Kid 24 closes - - no one else touches it.

Can muscle through (O = Open, C = Closed)

Locker #	1	2	3	4	5	6	7	8 ...
Kid 1	O	O	O	O	O	O	O	O ...
Kid 2	O	C	O	C	O	C	O	C ...
Kid 3	O	C	C	C	O	C	O	C ...
Kid 4	O	C	C	O	O	C	O	O ...
Kid 5	O	C	C	O	C	C	O	O ...
Kid 6	O	C	C	O	C	C	O	O ...
Kid 7	O	C	C	O	C	C	C	O ...
Kid 8	O	C	C	O	C	C	C	C ...

OR, look at factors of each number

Primes and composites (non-square) have even number of factors

Implication -- each ends up as things started

Squares have odd number of factors

Implication -- each of these opposite of as things started

That is, based on its factors, students 1, 2, 3, 4, 6, 12, and 24 will visit locker 24. If a similar expectation is provided with locker 25, students 1, 5, and 25 will visit that locker. Notice that there is an even number of factors for locker 24, meaning that after all students visit it, the door will be in the original position. On the other hand, since an odd number of students visit locker 25, its door will be in the opposite position from when the whole scene began.

Folklore describes a young man and a king's daughter. They were very much in love and wanted to marry. The king did not want the wedding to occur because the young man was not of the appropriate breeding, and besides, the young man did not possess the wealth to provide the lifestyle to which the girl had become accustomed. The young man was a clever fellow and offered the king an

Problem Solving

opportunity that was too good to pass up. The boy offered to go away if the king would be willing to give him some grain. The king would place one kernel of grain on a square of a checkerboard. The second square would contain two kernels of grain, the third four, and so on, until all 64 unit squares on the checkerboard were used. The total of the kernels of grain would be the young man's wealth and with it, he would leave, unless the king deemed otherwise. The king accepted the challenge. Do you think the young man married the king's daughter?

$2^{64} = 1.84 \times 10^{19}$ or 18,400,000,000,000,000,000 kernels of grain, just on the last square. The one before it would contain 9,220,000,000,000,000,000 kernels of grain. The young man ended up possessing more grain than the world could produce in several years.

A sheet of paper is folded in half, and then in half again, and again, and so on, until a total of 50 folds is made. The question is, "How high is the stack of paper?" The assumption here is that the 50 folds can be made.

Typical paper is 0.003 inches thick.

The stack of paper is 2^{50} thick \times 0.003 \approx 3.3776×10^{12} inches thick.

$$\text{Stack} \approx \frac{3.3776 \times 10^{12} \text{ inches}}{\left(\frac{12 \text{ inches}}{\text{foot}}\right)\left(\frac{5280 \text{ feet}}{\text{mile}}\right)} \approx 53,309,655 \text{ miles high!}$$

Use paper that is 0.004 inches thick and the stack is 70,901,841 miles high.

What is the largest number you can write using only three digits? The digits may be repeated. The first answer is typically 999, but after some thought you hear 99^9 or 9^{99} . This leads to a discussion as to which is larger. This is a wonderful opportunity to use rounding skills. We know 9 is close to 10 and 99 is close to 100, so we will round in both cases. Thus, 99^9 becomes 100^{10} and 9^{99} is close to 10^{100} . We know 100^{10} equals $(100)(100)(100) \dots$ ten times or 1 followed by 20 zeros. Similarly, 10^{100} is $(10)(10)(10) \dots$ one hundred times or 1 followed by 100 zeros. Thus, 9^{99} is larger than 99^9 , by quite a bit. But, the real answer is

9^{99} and the result is huge! The problem is worked from the top down, so the base 9 would be taken to the 387,420,000th power. The resultant value is about 500 miles long if typed using a 10-pitch font.

A customer enters a 7-11 convenience store and selects four items. The clerk informs the customer that the total cost of the four items is \$7.11 (excluding tax). The customer was amazed that the cost of the items was the same as the store name. The clerk informed the customer that the price of each item was multiplied to arrive at the total. The customer calmly informed the clerk that the prices should be added, not multiplied. The clerk obliged and the sum was \$7.11 for the same four items. What was the exact cost of each item (no rounding is necessary to obtain the exact values)? NOTE - - this proof is beyond many college students, but the problem is common. I list them both here mainly for your reference.

Problem Solving

7-11 - - This proof was developed by Dr. Howard Sherwood who is retired from the mathematics department at the University of Central Florida.

Find x , y , z , and w such that $(x)(y)(z)(w) = 7.11$ and $x+y+z+w = 7.11$, where x , y , z , and w are exact prices of objects.

Let $X=100x$, $Y=100y$, $Z=100z$, and $W=100w$.

Then X , Y , Z , and W are integers and

$$(X)(Y)(Z)(W) = (3^2)(79)(2^6)(5^6) \quad [1]$$

$$X+Y+Z+W = 711 \quad [2]$$

Since 79 is prime and 79 divides $(X)(Y)(Z)(W)$, we must have 79 divides one of X , Y , Z , or W .

Without loss of generalization, suppose 79 divides X . Then $X = 79(X_1)$ and [1] and [2] become

$$(X_1)(Y)(Z)(W) = (3^2)(79)(2^6)(5^6) \quad [3]$$

$$(X_1)+Y+Z+W = 711 \quad [4]$$

$1 \leq (X_1) < 9$ and (X_1) divides $(3^2)(2^6)(5^6)$ so $(X_1) = 1, 2, 3, 4, 5, 6, 7, \text{ or } 8$. (we will eliminate all but 4)

Suppose $(X_1) = 8$. Then [3] and [4] become

$$(Y)(Z)(W) = (3^2)(2^3)(5^6) \quad [5] \text{ and}$$

$$Y+Z+W = 79 \quad [6]$$

At least one of Y , Z , and W cannot be divisible by 5 otherwise 79 would be also. Thus 5^3 divides one of Y , Z , or W and [6] is violated. Thus, $(X_1) \neq 8$.

Suppose $(X_1) = 6$. Then [3] and [4] become

$$(Y)(Z)(W) = (3)(2^5)(5^6) \quad [7] \text{ and}$$

$$Y+Z+W = (7)(79) \quad [8]$$

Since 5 cannot divide one of Y , Z , or W , 5^3 must divide exactly two of them (because 5^4 would be too big a divisor for any one of them).

But then $Y+Z+W > (3)(79)$. Hence $(X_1) \neq 6$.

Suppose $(X_1) = 2$. Then [3] and [4] become

$$(Y)(Z)(W) = (3^2)(2^5)(5^6) \quad [9] \text{ and}$$

$$Y+Z+W = (7)(79) \quad [10]$$

Since 5 cannot be a divisor of every one of Y , X , and W and since $5^4 = 625 > 553 = (7)(79)$, we can see that 5^3 must be a divisor of exactly 2 of the factors. Without loss of generalization, let $Y = (5^3)(Y_1)$ and $Z = (5^3)(Z_1)$.

Then [9] and [10] become

$$(Y_1)(Z_1)(W) = (3^2)(2^5) \quad [11] \text{ and}$$

$$125(Y_1)+125(Z_1)+W=(7)(79) \quad [12].$$

Now, $(Y_1) + (Z_1) \leq 4$ otherwise [12] is false. If $(Y_1) + (Z_1) = 4$ then [12] becomes $W=53$ which contradicts [11].

Problem Solving

So, $(Y_1) + (Z_1) \neq 4$. If $(Y_1) + (Z_1) = 3$ then [12] becomes $W=178$, which contradicts [11]. So, $(Y_1) + (Z_1) \neq 3$. If $(Y_1) + (Z_1) = 2$, then [12] becomes $W=303$ which also contradicts [11]. Thus $(Y_1) + (Z_1) \neq 2$.

Suppose $(X_1) = 5$. Then [3] and [4] become
 $(Y)(Z)(W) = (3^2)(2^6)(5^5)$ [13] and
 $Y+Z+W = (4)(79)$ [14].

Since not all of Y , Z , and W can be divisible by 5 and none of them can be divisible by 5^4 , exactly one of them is divisible by 5^3 and exactly one of the others is divisible by 5^2 . Without loss of generality, let $Y=125(Y_1)$ and $Z=25(Z_1)$. Then [13] and [14] become

$(Y_1)(Z_1)(W) = (3^2)(2^6)$ [15] and
 $125(Y_1) + 25(Z_1) + W = (4)(79)$ [16].

Since $(4)(79)=316$ and $(125)(3)=375$, it is clear that $(Y_1)=1$ or $(Y_1)=2$. If $(Y_1)=1$, then $25(Z_1)+W=191$ whence exactly one of (Z_1) and W is odd which in light of [15] would say that either 64 divides (Z_1) or 64 divides W . Clearly 64 does not divide (Z_1) because then $25(Z_1)+W > 191$. Thus, 64 divides W but then $(Z_1) = 1, 3, \text{ or } 9$ which leads to a contradiction. Thus, $(Y_1) \neq 1$.

So, $(Y_1) = 2$. But then [15] and [16] become
 $(Z_1)(W) = (3^2)(2^5)$ [17] and
 $25(Z_1) + W = (2)(3)(11)$ [18]

From [17] and [18] it is clear that (Z_1) and W are both even. But $(Z_1) < 4$ so $(Z_1) = 2$ whence $W=(3^2)(5^4)$ and $W=(2)(3)(11) - (25)(2)=16$, a contradiction. Thus, $(Y_1) \neq 2$ either, and in turn, $(X_1) \neq 5$.

Suppose $(X_1) = 1$. Then from [3] and [4] we get
 $(Y)(Z)(W) = (3^2)(2^6)(5^6)$ [19] and
 $Y+Z+W=(2^3)(79)$ [20]

Clearly 5^3 must divide one of Y , Z , or W , but 5^4 cannot divide any of Y , Z , and W because if it did, the left side of [20] would be too big to equal $(2^3)(79)$. Without loss of generality we may write Y as $125(Y_1)$. Now 5 cannot divide each of Y , Z , and W so 5^3 must divide one of Z or W . Without loss of generality we write $Z=125(Z_1)$. Then [19] and [20] become

$(Y_1)(Z_1)(W) = (3^2)(2^6)$ [21] and
 $125(Y_1)+125(Z_1)+W=(2^2)(79)$. [22]

Since $\frac{(2^3)(79)}{125} = 5.056$, $(Y_1)+(Z_1) \leq 5$. But if $(Y_1)+(Z_1) = 5$, then $W=7$ which in light of [21], is also impossible.

If $(Y_1)+(Z_1) = 4$, then $W=132$ which in light of [21], is also impossible.

If $(Y_1)+(Z_1) = 3$, then $W=257$ which in light of [21], is also impossible.

Problem Solving

If $(Y_1)+(Z_1) = 2$, then $W=382$ which in light of [21], is also impossible.
Hence $(X_1) \neq 1$.

Suppose $(X_1) = 3$. Then from [3] and [4] we get

$$(Y)(Z)(W)=(3)(2^6)(5^6) \quad [23] \text{ and}$$

$$Y+Z+W=(2)(3)(79) \quad [24].$$

Once again 5^3 divides one of Y , Z , or W but 5^4 does not divide any of them, so without loss of generality we may assume $Y=125(Y_1)$ and $Z=125(Z_1)$. Hence [23] and [24] become

$$(Y_1)(Z_1)(W)=(3)(2^6) \quad [25] \text{ and}$$

$$125(Y_1)+125(Z_1)+W=(2)(3)(79) \quad [26]$$

Since $\frac{(2)(3)(79)}{125} = 3.792$, we see that $(Y_1)+(Z_1) \leq 3$. If $(Y_1)+(Z_1) = 3$, then

[26] says $W=99$ which contradicts [25]. If $(Y_1)+(Z_1) = 2$, then [26] says $W = 224 = (32)(7)$, which contradicts [25].

Thus, $(X_1) \neq 3$.

Finally we have shown that $(X_1)=4$ and [3] and [4] become

$$(Y)(Z)(W)=(3^2)(2^4)(5^6) \quad [27] \text{ and}$$

$$Y+Z+W=(5)(79). \quad [28]$$

Since $5^4 > (5)(79)$, it is clear that 5^4 does not divide any of Y , Z , or W . Next 5^3 cannot divide exactly 2 of Y , Z , or W , otherwise from [28] we would see that 5 would divide the remaining one contradicting [27]. Thus 5^3 divides one of Y , Z , or W , 5^2 divides one of the remaining two, and 5 divides the third. Without loss of generality we obtain $Y=125(Y_1)$, $Z=25(Z_1)$, and $W=5(W_1)$. Thus [27] and [28] become

$$(Y_1)(Z_1)(W_1) = (3^2)(2^4) \quad [29] \text{ and}$$

$$25(Y_1)+5(Z_1)+ (W_1) = 79 \quad [30]$$

Since $\frac{79}{25} = 3.16$, $(Y_1) \leq 3$. Clearly $(Y_1) \neq$ otherwise [30] becomes

$5(Z_1)+(W_1)=4$ which is impossible. Thus $(Y_1) \leq 2$. If $(Y_1) = 2$, then

$$(Z_1)(W_1) = (3^2)(2^3) \quad [31] \text{ and}$$

$$5(Z_1) + (W_1) = 29 \quad [32].$$

The second of which tells us that exactly one of (Z_1) or (W_1) is even whence 8 divides (Z_1) or 8 divides (W_1) . But 8 does not divide (Z_1) because of [32]. If 8 divides (W_1) , then $(Z_1) = 1, 3, \text{ or } 9$, none of which is possible. Thus, $(Y_1) = 1$. Now, [29] and [30] become

$$(Z_1)(W_1) = (3^2)(2^4) \quad [33] \text{ and}$$

$$5(Z_1) + (W_1) = (2)(3^3) \quad [34].$$

Problem Solving

Now 2^2 cannot divide both (Z_1) and (W_1) because of [34]. But 2 must divide each of (Z_1) and (W_1) otherwise we would contradict [33] and [34]. Thus 2 divides one of (Z_1) and (W_1) and 2^3 divides the other.

Suppose $(Z_1) = (2^3)(Z_2)$ and $(W_1) = 2(W_2)$. Then [33] and [34] become
 $(Z_2)(W_2) = (3^2)$ [35] and
 $20(Z_2) + (W_2) = (3^3)$ [36] which is impossible. Thus (2^3) cannot divide (Z_1) . Hence (2^3) divides (W_1) and 2 divides (Z_1) . So we may write $(Z_1) = 2(Z_2)$ and $(W_1) = (2^3)(W_2)$. [35] and [36] become

$$(Z_2)(W_2) = (3^2) \quad [37] \text{ and}$$

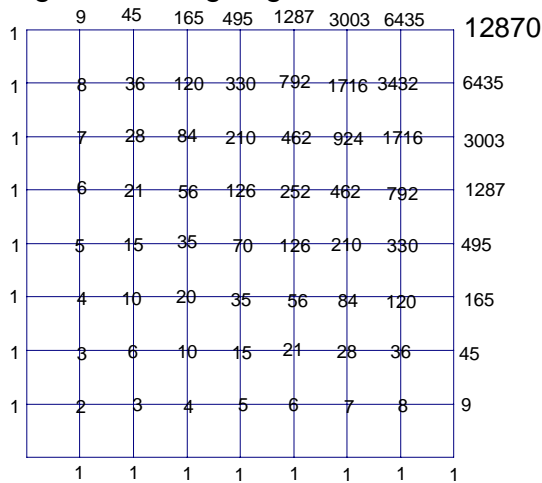
$$5(Z_2) + 4(W_2) = (3^3) \quad [38].$$

From this it follows that $(Z_2) = (W_2) = 3$. Thus
 $X = (4)(79) = 316,$
 $Y = (125)(1) = 125,$
 $Z = (5^2)(2)(3) = 150, \text{ and}$
 $W = (5)(2^3)(3) = 1.20.$

And, in turn,
 $x = \$3.16,$
 $y = \$1.25,$
 $z = \$1.50, \text{ and}$
 $w = \$1.20.$

Dr. Howard Sherwood (retired from the mathematics department at the University of Central Florida) developed this proof.

How many different routes can be taken to go from the bottom left corner of a checkerboard to the upper right corner when you may move only up or to the right along the existing segments between the squares?



The numeral at each intersection indicates the number of routes that can be taken to get to that point.

Problem Solving

Resources

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