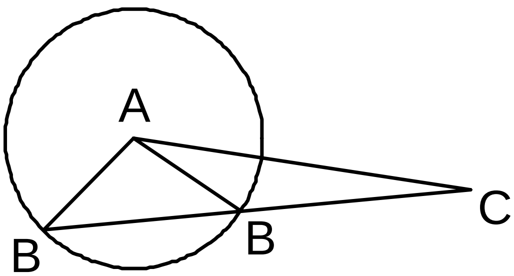
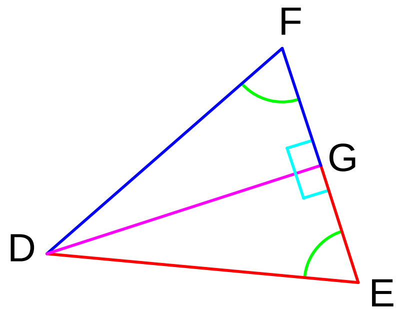
“Mathematics does not grow through a monotonous increase of the number of indubitably established theorems but through the incessant improvement of guesses by speculation and criticism, by the logic of proofs and refutations” (Lakatos, 1976, p. 5). We know there is a need to have proofs in the study of mathematics. Without them, we may arrive at incorrect conclusions. In geometry, it could be conjectured that Side, Side, Angle could be used to prove two triangles congruent. Initial considerations might generate the impression that this is a reasonable assumption, particularly if several other side and angle (ASA, SAS, etc.) combinations have been investigated and shown to be true. But, a construction can be used to explain how two different triangles can be generated given sides AB and AC and angle BCA. Different locations of segment AB yield different lengths for side BC, contradicting congruency. The approach is flawed because the question becomes which side AB is the one needed?



The Standards (NCTM, 1989) suggest a change in the role of proof in our mathematics curriculum. The call is for a decrease in attention given to Euclidean geometry as an axiomatic system and two-column proofs. At the same time, it is recommended that short sequences of theorems be developed and that deductive arguments be expressed orally and in paragraph or sentence form.

Suppose the desire is to prove the base angles of an isosceles triangle are congruent using the typical two-column proof as well as paragraph form. You are given triangle DEF with DE ≅ DF. Prove ∠DEF ≅ ∠DFE.



DEF is a triangle	Given
DE ≅ DF	Given
DG ⊥ EF	Construction
∠DGF = rt. ∠	Dfn. ⊥ lines
ΔDGF is rt.	Dfn. rt. Δ
∠DGE = rt. ∠	Dfn. ⊥ lines
ΔDGE is rt.	Dfn. rt. Δ
∠DGE ≅ ∠DGF	Trans. property of =
DG ≅ DG	Reflx. property of =
ΔDGE ≅ ΔDGF	Hypot., leg of rt. Δ
∠DEF ≅ ∠DFE	Dfn. ≅

Other two-column methods that could be used to conclude that ∠DEF ≅ ∠DFE include bisecting angle EDF or locating the midpoint of side EF.

Another less common method can be used in the form of a paragraph proof.

> You are given triangle DEF with sides DE and DF congruent. You are to prove angle DEF congruent to angle DFE. Construct triangle DEF. Imagine lifting the triangle, flipping it, and placing it back on top of itself so DE is on top of DF and DF is on top of DE. You now have triangle DEF on top of triangle DFE with all points matching. This shows DE congruent to DF and DF congruent to DE. EF is congruent to itself. The two triangles are congruent by Side, Side, Side. Since the triangles are congruent, ∠DEF ≅ ∠DFE.

This can be done via technology with software capable of dealing with transformations.

The change in emphasis on proof is based largely on information about learning patterns of students. Mathematical proof requires an understanding of definitions and logic, but also depends on insight into why and how they work and connect. Before we can expect students to effectively prove things, we must help them become independent thinkers, understanding the need for precision of language, definition, and expression.

WHAT IS PROOF?

Proof convinces one that a discourse and its associated conclusion are factual. Proven things could be replicated and do not contradict something else known to be true. Depending on the setting, the dialogue takes on a variety of approaches. A young child, talking with another, may attest to the proof of something being true because “someone said so.” Here, the child yields to authority. Eventually, as the youngster gets a little older, the route to proof might turn to a less acceptable process: something like saying, “I am tougher than you are and will beat you up to prove it.” The associated fisticuffs, although undesirable, do show an advancement in the development of proof. Students see lawyers convincing, beyond a reasonable doubt, that the position of their client should prevail in court - - a proof of sorts.

Eventually the need for more exactness in proof develops. An immediate question becomes how to express the proof. Regretfully, proof of any kind does not appear very often in mathematical instruction outside of geometry. A structured approach to establishing the validity of an argument follows a clearly established sequence of steps leading to a result that is checked to assure the correctness of the answer. The manner used for the proof is not critical. What is most important is the fact that a coherent, logically correct discussion is presented.

CONVINCING STUDENTS PROOF IS NEEDED

The discussion about using SSA as a congruence theorem is a beginning point to establish the need for proof. Without investigation and verification or denial of ideas, we have no way of determining what is and is not acceptable thinking. Much of school mathematics is delivered in a manner that implies it was chipped in stone and handed down by some mathematical genius. We know that is not the case. How do we convince students of the need for proof?

One way would be to create situations that appear to be true but, in reality, are not. Lead the students to make false conjectures and then help them see the error of their ways. An example can come from the following pair of equations that are to be solved for the point of intersection. This example is particularly well placed when solution by substitution is being covered. The equations are: (1) $y - 3x = 10$ and (2) $x = \frac{(y + 2)}{3}$. Substituting for x in (1), $y - \frac{3(y + 2)}{3} = 10$ or $y - y - 2 = 10$, which yields that $^{-}2 = 10$. Ridiculous! But, what went wrong? There is a need for further investigation in order to determine the error. Typically the first approach will be to redo the arithmetic. That provides no clue. A lesson from problem solving where alternate avenues were suggested as a means of strengthening skills will be helpful. One alternate route to resolving the issue would be to graph these two equations and they end up being parallel lines, so there is not intersection.

A classic problem involves asking a class if they think two people in the group will share a birthday (both month and day of month). The usual response is that such an event is highly unlikely. Common logic says it normally would take a large group before there would be at least one matching month and day.

Use 365 for the number of days in a year. The month and day of birth for the first person is announced. There are then 364 days left that do not match. If a second person enters consideration (and does not match the first date), there would be 363 days left that would not match. The probability of a third person not matching means that this third person has 363 options left. At this stage of development, $\frac{(365)(364)(363)}{365^3}$ expresses the probability of having no match, which in this case happens to be 0.9917958. If 10 people are considered, the probability is 0.8830518 that none of their birth dates will be the same, and for 20, the probability is 0.5885616. Extending to 23 people, the probability is 0.4927027 that no 2 people will have a matching birthday. Another way of saying this is there is a 0.5072973 probability that, in a group of 23 people, at least 2 of them will share a birthday (only in terms of day of the month and the month itself). Preposterous! That just does not seem right! In a group of 23 people, there is a fifty-fifty chance that two of them will share a birthday. Do the activity with a group and see what happens. Even if the group used does not have a match, the question still lingers for many - - “How or why?” The question is generated even more quickly and emphatically if there are 2 people with the same birthday in the group of 30.

As this problem has been done with students, they almost always wanted to know how such a seemingly impossible answer could happen. They wanted to prove why it was right or wrong. The “right” environment creates the need for a proof.

One group of students concluded that the probability of 2 people not having the same birthday would be $\frac{365!}{(363!)(365^2)}$ = 0.9973 (note this is a different expression of the earlier stated approach). Similarly, the probability of 3 people not having the same birthday is $\frac{365!}{(362!)(365^3)}$ = 0.9918. They developed a general formula for the probability of not having the same birthday $\frac{365!}{((365 - n)!)(365^n)}$, where n is the number of people.

The class decided to use a spreadsheet to summarize things.

Number Of People	Probability Of 2 With Different Birthdays	Probability Of 2 With Same Birthday
2	0.9973	0.0027
3	0.9918	0.0082
4	0.9836	0.0164
5	0.9729	0.0217
6	0.9595	0.0405
7	0.9438	0.0562
8	0.9257	0.0743
9	0.9054	0.0946
10	0.8831	0.1169
20	0.5886	0.4114
21	0.5563	0.4437
22	0.5243	0.4657
23	0.4927	0.5073
24	0.4617	0.5383
25	0.4313	0.5687
30	0.2937	0.7063
40	0.1088	0.8912
50	0.0342	0.9704
60	0.0058	0.9942

Sometimes a situation appears to prove something is true. The following description is of a scheme that is illegal but it does show the need for an explanation (proof). A financial advisor sent out 64,000 letters to potential clients.

Of those letters, 32,000 said the value of a stock would rise over a given period. The content of the other 32,000 letters said the value of that same stock would not rise (implying it would stay the same or fall) over the same period.

After a time, a second round of letters was sent, but only to individuals who had received an original letter containing a prediction that had been correct. The same stock, or a different one, could have been used in the second letter. The contents of the letter would be the same as the first one, with the possible exception of different stock. This time, of the letters sent, 16,000 predicted the stock would rise in value and 16,000 stated it would not. At the end of this second period, a quarter of the original 64,000 people, or 16,000 had received two letters with correct predictions. The scenario was repeated two more times and the original group of 64,000 was reduced to 4,000 individuals who had received four letters correctly predicting the movement of the stock market.
A fifth letter was sent, but this one did not contain a prognostication. Instead, it carried a simple explanation and reminder that the correct movement of the stock market had been demonstrated in the last four letters. That succession of correct forecasts “proved” to the individual that the path stocks take could be predicted. This letter carefully explained that the service would be made available only to a select few individuals for a modest annual fee. The implication was that no matter what amount was charged for the service, the client would easily recoup the expenses from profits.

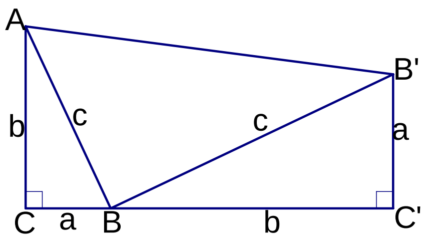
Suppose, for the sake of this discussion, the letter, including letterhead, envelope, software, hardware, and postage required to produce it, was sent for $0.50. The payment for the producer’s time (not jail time here, but the time spent bringing this scheme to fruition) will be computed later. The 128,000 (64,000 + 32,000 + 16,000 + 8,000 + 4,000) letters sent generated an expense of $64,000. Suppose each potential client was charged an annual fee of $500 for the services. If each of the individuals join, $1,936,000 was generated over a short period of time. Delete the $64,000 cost and the “profit” is still rather significant. Even if only half of the potential clients sign up, after taking out the costs, the profit was close to $900,000. Only 128 people paying $500 cover the expenses. Anything above that is profit. Do you think there would be more than 128 takers on a scheme like this? Do you see why it is illegal? If you were to do it, you would be classified as doing business with the intent of defrauding people. This is one of the classic con games that has been used on unsuspecting victims who are duped into believing there are “foolproof” methods by which to get rich quick. Do you also see where “proof” can be a deceptive concept?

“Seeing is believing” is often accepted as proof. Paulos (1988) described firewalking on a bed of hot coals. When an individual performs this feat (pun intended), the discussion centers on the ability to control the mind and exclude the associated pain that should go with walking on hot coals. The walk is observed and spectators are convinced that the described process of mind over matter works because of what they have seen. The event is based on a little-known fact that dehydrated wood has an extremely low heat content and is a very poor conductor of heat. A person walking quickly across the coals feels little of the apparent heat. Of course, the mind-over-matter routine sounds much more glamorous and once again opens the door to opportunity for defrauding individuals. Seeing is not always believing, and if something seems too good to be true, it probably is. Investigation will yield a “proof.”

PROOFS IN THE HISTORY OF MATHEMATICS

Mathematics history grows on proofs. Singling out only a few examples here cannot do justice to all the developments. Perhaps that is one reason why the study of the history of mathematics and inserting it throughout the curriculum is encouraged.

James Abram Garfield, 26th President of the United States, is credited by some with one of the many proofs of the Pythagorean theorem. Garfield created right triangle ABC.



He duplicated it and had AC and B'C' both perpendicular to segment CC', creating a trapezoid. He used the idea that the whole is the sum of the parts and expressed the area of the trapezoid in terms of the three triangles. He also used the formula for the area of a trapezoid to establish the equation:

$$\frac{ab}{2} + \frac{ab}{2} + \frac{c^2}{2} = \frac{(a + b)(a + b)}{2}$$

$$ab + \frac{c^2}{2} = \frac{(a + b)(a + b)}{2}$$

$$2ab + c^2 = (a + b)(a + b)$$

$$2ab + c^2 = a^2 + 2ab + b^2$$

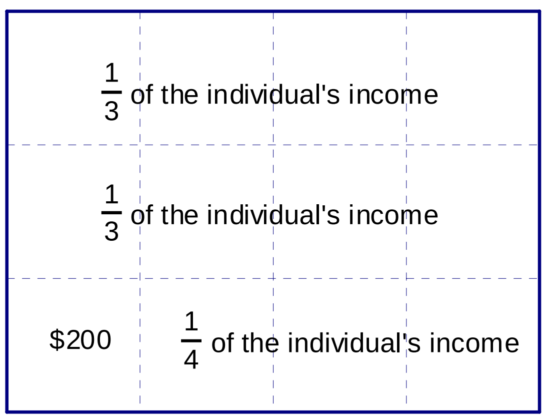
$$c^2 = a^2 + b^2$$

The Garfield approach to “proving” the Pythagorean theorem has the added advantage of providing an avenue to connect the study of mathematics to social science.

PROOFS IN DIFFERENT ENVIRONMENTS

Algebra students are accustomed to using their basic skills to do problems. They need to be convinced that there may be acceptable alternative solutions to a problem. A typical algebra textbook problem could be: “Suppose one-third of an individual’s income is spent on housing, another third is spent on transportation and education, and 25% on food and entertainment. The individual has $200 left for saving, giving, investing, and shopping? How much does the individual make a month? Algebraically, a solution could be determined through the equation $\frac{x}{3} + \frac{x}{3} + \frac{x}{4} = x - \$200$, where x represents the monthly income. Solving, $\$200 = \frac{x}{12}$, which gives that x = $2400.

This same algebraic problem can be solved geometrically by having the total income (big rectangle) divided into thirds horizontally and fourths vertically. The net result is a total of 12 congruent squares, four of which represent a third and three of which represent a fourth. There is one square left after the fourth and both thirds are represented. But, we know that the left over money is $200. Since there are 12 congruent squares, the total income must be 12 x $200 = $2400.



Proof is not always direct manipulation that verifies the issue in question. Mathematical induction occurs in a variety of settings throughout the curriculum. Mark Twain in Life on the Mississippi (Adler, 1972, Book 2, pp. 56–58.) described a series of events involving the Mississippi River abandoning a course and acquiring a new, shorter one when considering the distance traveled between Cairo, Illinois, and New Orleans, Louisiana:

> The water cuts the alluvial banks of the “lower” river into deep horseshoe curves; so deep, indeed, that in some places if you were to get ashore at one extremity of the horseshoe and walk across the neck, half or three-quarters of a mile, you could sit down and

rest a couple of hours while your steamer was coming around the long elbow at a speed of ten miles an hour to take you on board again. When the river is rising fast, some scoundrel whose plantation is back in the country, and therefore of inferior value, has only to watch his chance, cut a little gutter across the narrow neck of land some dark night, and turn the water into it, and in a wonderfully short time a miracle has happened: to wit, the whole Mississippi has taken possession of that little ditch, and placed the countryman’s plantation on its bank (quadrupling its value), and that other party’s formerly valuable plantation finds itself away out yonder on a big island; the old watercourse around it will soon shoal up, boats cannot approach within ten miles of it, and down goes its value to a fourth of its former worth. Watches are kept on those narrow necks at needful times, and if a man happens to be caught cutting a ditch across them, the chances are all against his ever having another opportunity to cut a ditch. . . . Since my own day on the Mississippi, cutoffs have been made at Hurricane Island, at Island 100, at Napoleon, Ark., at Walnut Bend, and at Council Bend. These shortened the river, in the aggregate, sixty-seven miles. In my own time a cut-off was made at American Bend, which shortened the river ten miles or more.

Therefore the Mississippi between Cairo and New Orleans was twelve hundred and fifteen miles long one hundred and seventy-six years ago. It was eleven hundred and eighty after the cut-off of 1722. It was one thousand and forty after the American Bend cut-off. It has lost sixty-seven miles since. Consequently, its length is only nine hundred and seventy-three miles at present.

Now, if I wanted to be one of those ponderous scientific people, and “let on” to prove what had occurred in the remote past, by what had occurred in a given time in the recent past, or what will occur in the far future by what has occurred in late years, what an opportunity is here! Geology never had such a chance, nor such exact data to argue from! Nor “development of species,” either! Glacial epochs are great things, but they are vague—vague. Please observe: In the space of one hundred and seventy-six years the Lower Mississippi has shortened itself two hundred and forty-two miles. That is an average of a trifle over one mile and a third per year. Therefore, any calm person, who is not blind or idiotic, can see that in the Old Oolitic Silurian Period, just a million years ago next November, the Lower Mississippi River was upward of one million three hundred thousand miles long, and stuck out over the Gulf of Mexico like a fishing-rod. And, by the same token any person can see that seven hundred and forty-two years from now the lower Mississippi will be only a mile and three-quarters long,

> and Cairo and New Orleans will have joined their streets together, and be plodding comfortably along under a single mayor and a mutual board of aldermen. There is something fascinating about science. One gets such wholesale returns of conjecture out of such a trifling investment of fact.

The foundations for proof can be established by examining divisibility rules. Most of the proofs for divisibility rules depend on an understanding of expanded notation and the idea that if a number is divisible by some factor, then any multiple of that number will also be divisible by that same factor. For example, show 27 as 3 + 9 + 15. The common element, 3, can be factored out and the number written as the product of that factor and the sum of the residue of each of the terms, 3(1 + 3 + 5). This shows the original number is divisible by a common factor. This concept is difficult for many students to comprehend, and it may take some time for them to be comfortable with the idea. Once the realization occurs, students can easily understand, and produce, proofs for divisibility rules.

What is the rule for showing a number to be divisible by 2? Typically this question is answered with statements like, “The last digit is even” or “The last digit is a 2, 4, 6, 8, or 0.” Either one of those is acceptable, along with others, but the real question is, “How do you know that to be true?” or “Can you prove it?” Extended investigation generally reveals that most people were given the rule and accepted it as gospel, never paying any attention to whether or not there was a need to prove the statement true. One of the goals of NCTM’s Standards is communication, which involves explaining why things work, implying proof. Discussion for proving divisibility by 2 often becomes cyclic, and students struggle for a way to begin. Consider any number Xy where X is any integer and y is any digit. In 7,358 X is 735 and y is 8. (A negative 7,358 would yield the same idea, but it would be hampered by the thinking associated with positive and negative. Thus, for the sake of simplicity, positive is used.) Xy can be written in expanded notation as (X)(10) + y. No matter what integer is used for X, (X)(10) must be divisible by 2, because 10 is always divisible by 2. Any multiple of 10 will also be divisible by 2. One term of the expanded form is guaranteed to be divisible by 2. If the other term is also divisible by 2, a 2 can be factored out of the expanded form and the original number written as 2 times something. This verifies that the original number is divisible by 2. But, when can a 2 be factored out of both terms? Only when y is even, or 0, 2, 4, 6, or 8, which gives the rule statement and the proof is completed.

The proof for divisibility by 5 and even 10 is similar to that for 2, except that the y must be either 5 or 0 for divisibility by 5 and only 0 for divisibility by 10. It should not be too difficult for students to produce these proofs once they have seen the one for 2.

Divisibility by 4 is proven by applying the “2 rule” twice, or in a manner similar to that of 2, 5, and 10, with one notable exception. The expanded expression of the number being considered cannot be (X)(10) + y because 10 is not divisible by 4. The first counting number power of 10 that is divisible by 4 is 10^2 or 100. So, the number being considered needs to be written as Xyz where X is any integer and y and z are any digits. The number is then written as (X)(100) + yz (note y and z are not multiplied here - - they represent the tens and ones digits of the number). Because 100 is divisible by 4, attention is turned to the last two digits, yz. If they are divisible by 4, the whole number is. If yz is not divisible by 4, then the initial number is not, because 4 cannot be factored out of both terms. For example,

5,732 = 57(100) + 32
= 57 (4)(25) + 4 (8)
= 4[57(25) + 8].

If the initial number is 5,731, four cannot be factored out of 31. Thus, 5,731 is not a multiple of 4.

Once divisibility for 4 has been established, 8 can be considered in a similar manner. A rule for 16 can be established as well, but there is some question as to the value of it, due to the magnitude of the numbers involved. As the proofs of divisibility for 2, 4, 8, 16, are done, notice the parallel between the exponents of 2 and 10 (2^1 and 10^1); 2^2 and 10^2 for divisibility by 4; 2^3 and 10^3 for divisibility by 8; 2^4 and 10^4 for divisibility by 16. Patterns abound in mathematics.

The proofs of divisibility by counting number powers of 3 use the idea of expanded notation along with that of having a common factor in each term so the original number can be expressed as a multiple of a counting number power of 3. Suppose you want to know if the three-digit number XYZ, where X, Y, and Z are all digits, is divisible by 3. The rule says to find the sum of the digits. If that sum is divisible by 3, then the original number is also. The “3 rule” could be repeated on the new sum, if desired, before determining divisibility.

Mathematicians often use the known answer as a clue for how to proceed with a proof. Rewrite XYZ using expanded notation as (X)(100) + (Y)(10) + Z. This approach shows a useful proof technique. The answer is known; we want to have the sum of the digits as a part of expressing XYZ. Using that information, we have to rewrite XYZ somehow so “X + Y + Z” are some of the terms (perhaps not grouped together as shown here). Focus on (X)(100) in the expanded form of XYZ, because once it is seen how to rewrite this, the rest are similar. The challenge is to find a way to rewrite (X)(100) so it is a sum of X and something. Using the clue of knowing one term of the final answer needs to be X, we have to express 100 in a manner that will give us an X when we are all done. If the 100 were rewritten so (X)(100) becomes (X)(99 + 1) and distributing the X yields (X)(99) + (X)(1) or 99X + X. The X is now isolated. The 99 is a multiple of 3 and always divisible by 3. Using the same technique for (Y)(10),

XYZ = (X)(100) + (Y)(10) + Z
= (X)(99 + 1) + (Y)(9 + 1) + Z
= (X)(99) + (X)(1) + (Y)(9) + (Y)(1) + Z
= 99X + 9Y + X + Y + Z.
It is known that 99X and 9Y will always be divisible by 3, assuring the ability to factor 3 out of those two terms. The only thing left to consider is X + Y + Z, which is the sum of the digits. If the sum X + Y + Z is divisible by 3, then the original number can be expressed as a multiple of 3, which would be factored out of XYZ. If the sum of the digits was not a multiple of 3, the original number cannot be expressed as a multiple of 3.

Divisibility by 9 works exactly like that of 3 except that the sum of the digits must be divisible by 9.

Divisibility by 6 uses a combination of the 2 and 3 rules. The easiest way to explain the 6 rule is to check to see if the number in question is even. If it is not, the number cannot possibly be divisible by 6. If the number in question is even, then apply the 3 rule to determine if it is divisible by 6. The discussion about why divisibility for 6 uses the 2 and 3 rules should include items dealing with the prime factorization of 6.

Divisibility by 11 seems complex at first glance. Closer investigation shows that the proof is relatively simple, revolving around renaming 10 as 11 - 1 and using the concept of expanded notation. Consider $(11 - 1)^2$. Expanding yields $11^2 - 2(11)(1) + 1^2$. Because 11 is a factor of two of the terms in the expansion, that sum must be divisible by 11. Expanding $(11 - 1)^3$ gives all terms except 1^3 as a multiple of 11. The situation is similar for all cases of $(11 - 1)^n$, where n is any counting number - - all terms except for 1^n will be multiples of 11. The sign of 1^n will be negative for odd values of n, and positive for even ones.

Using this information to check for divisibility by 11, on the number UVWXYZ where U, V, W, X, Y, and Z are any digits, UVWXYZ would be rewritten as $U(10)^5 + V(10)^4 + W(10)^3 + X(10)^2 + Y(10)^1 + Z(10)^0$ or as $U(11 - 1)^5 + V(11 - 1)^4 + W(11 - 1)^3 + X(11 - 1)^2 + Y(11 - 1)^1 + Z(11 - 1)^0$. From the earlier discussion, expansion of this polynomial will yield a set of terms that are multiples of 11 plus some residue. We know most terms are multiples of 11. The only terms in question are ^-U, ^+V, ^-W, ^+X, ^-Y, and ^+Z. Inspection shows the rule to be the rightmost digit minus its left neighbor, plus the next left neighbor, and so on, until all digits are considered. In other words, the sum of every other digit is subtracted from the sum of the rest of the digits (for divisibility, the sign is not important). Only if that missing addend is a multiple of 11 will the original number be divisible by 11.

PROOFS STUDENTS ASK FOR

A typical question is, “Why is anything to the zero power one?” Wording is important. As asked, the question is not reflective of what is true because, for example, $0^0 \neq 1$. Excluding situations that are not true, establishing the fact that $x^0 = 1$ is relatively easy if the student has experience with exponents and the idea that anything nonzero and non-infinite divided by itself is one. $\frac{x^n}{x^n} = 1$ where n is any real nonzero, non-infinite value. But, $\frac{x^n}{x^n} = x^{n - n}$ or x^0. By the transitive property of equality, it must be the case that $x^0 = 1$.

Another example that is extremely perplexing to students involves repeating decimals. Even when they see a “proof” that $0.\overline{999} = 1$, they resist accepting it. The typical development would include a discussion that $0.\overline{999}$ is a repeating decimal that never ends. It would be distinguished from 0.9, 0.99, 0.999, and so on, each of which terminates and, thus, can be expressed as $\frac{9}{10}$, $\frac{99}{100}$, $\frac{999}{1000}$, and so on, respectively. Assuming the students possess some algebraic background and are able to subtract one equation from another, consider the following approach to why $0.\overline{999} = 1$.

Let $x = 0.\overline{999}$		(1) Given
$10x = 9.\overline{999}$		(2) Multiply (1) by 10
Subtract (1) from (2)		
	$10x = 9.\overline{999}$	
	$-\ \ x = 0.\overline{999}$	(3) Subtracting (1) from (2)
	$9x = 9.000$	(4) Collect like terms
	$x = 1$	(5) Divide (4) by 9
But,	$x = 0.\overline{999}$	Given
So,	$1 = 0.\overline{999}$	Transitive property of =

Assure students that the 9s repeat in $0.\overline{999}$ and in $9.\overline{999}$ as far as they want, and usually they will accept the subtracting of 9 from 9 in any place value to the right of the decimal, no matter how far to the right it is. This brings them to the conclusion that the missing addend for $9.\overline{999} - 0.\overline{999}$ is 9, which is generally accepted with little resistance. Only when the division is completed in Equation (5) and it is concluded that $1 = 0.\overline{999}$ does the resistance begin to build.

When students resist $0.\overline{999} = 1$, have the class try $2.\overline{444} + 3.\overline{555}$. Find the equivalent fractions for each addend:

Let n = $2.\overline{444}$ and then 10n = $24.\overline{444}$. Using the aforementioned procedure for $0.\overline{999}$ = 1, 9n = 22 and n = $\frac{22}{9}$. Similarly, let m = $3.\overline{555}$ and then 10m = $35.\overline{555}$. Subtracting as before gives m = $\frac{32}{9}$. Then $2.\overline{444} + 3.\overline{555}$ becomes $\frac{22}{9} + \frac{32}{9} =$ $\frac{54}{9}$ = 6 and so
$2.\overline{444} + 3.\overline{555} = 6$.

Number tricks provide opportunities for getting students to ask for a proof. A typical example would be, “We are going to add five two-digit numbers. You pick two addends, I pick three, and the sum will be 245.” In addition to getting students to ask how it works, this is a marvelous chance to have them practice addition skills and ask for more, without even knowing they are practicing. Limitations, like not permitting students to select a value that has the same digits, can be established initially. Suppose 34 is one of their numbers and 75 is the other. You would select 65 to be paired with 34 and 24 to be paired with 75, without telling them what you are doing. Those two pairs have a sum of 99, each of which can be expressed as 100 - 1. The four addends chosen so far have a sum of 200 - 2. The remaining addend, or “magic number,” is selected so that when 2 is subtracted from it, the missing addend will be, in this case, 45. This “magic number” is 47. This value will vary depending on the announced sum. Each “99” requires two addends. If the sum for this trick was 345, there would be a total of seven addends. Six of the addends would be for the three pairs of addends that would yield a total of 300 - 3. The last addend would be the “magic number,” which in this case would be 48, because 3 is to be subtracted from it. This trick can be extended to deal with addends having any number of digits in them. No matter what size the addends, a careful sequencing of problems can be used to lead students to conclude how the trick works.

CONCLUSION

Should we continue proving? Is there room for or a need for proofs at all levels of mathematics? Can students be convinced of a need to prove things? What is the role of proof in the mathematics curriculum today?

Our job, as teachers, is to entice our students to think. Their reasoning should not yield to authority, but rather reflect consideration about a given topic. They need to convince themselves of a position to be held, and they need to be able to support their position with reasons.

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